Value Distribution of L-Functions with Rational Moving Targets

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ABSTRACT

We prove some value-distribution results for a class of L-functions with rational moving targets. The class contains Selberg class, as well as the Riemann-zeta function.

Keywords: Value Distribution; Moving Target; L-Function; Selberg Class

1. Introduction

We define the class \( \mathcal{M} \) to be the collection of functions
\[
L(s) = \sum_{n=1}^{\infty} a(n)/n^s,
\]
satisfying Ramanujan hypothesis, Analytic continuation and Functional equation. We also denote the degree of a function \( L \in \mathcal{M} \) by \( d_L \) which is a non-negative real number. We refer the reader to Chapter six of [1] for a complete definitions. Obviously, the class \( \mathcal{M} \) contains the Selberg class. Also every function in the class \( \mathcal{M} \) is an L-function and the Riemann-zeta function is in the class. In this paper, we prove a value-distribution theorem for the class \( \mathcal{M} \) with rational moving targets. The theorem generalizes the value-distribution results in Chapter seven of [1] from fixed targets to moving targets.

Theorem. Assume that \( L \in \mathcal{M} \) and \( R \) is a rational function with \( \lim_{s \to \infty} R(s) \neq 1 \). Let the roots of the equation \( L(s) - R(s) = 0 \) be denoted by \( \rho_{\beta} = \beta + iT \). Then

(I) For any \( b > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\} \),
\[
\sum_{\beta \leq \pi T} (\beta - b) = O(T), \quad \text{as } T \to \infty.
\]

(II) For sufficiently large negative \( b \),
\[
2\pi \sum_{T \leq \gamma \leq 2T} (\beta - b) = (-b) d_L T \log \frac{4T}{e} + O(\log T),
\]
as \( T \to \infty \).

Proof of (I). It is known that if \( L \in \mathcal{M}, \) then
\[
L(s) = \sum_{n=1}^{\infty} a(n)/n^s = 1 + O(k_0^\sigma), \quad \text{as } \sigma \to \infty;
\]
where \( k_0 \) is the index of the first non-zero term of the sequence of \( \{a(n)\}_{n=1}^{\infty}, \) \( s = \sigma + it \) with \( \sigma, t \in \mathbb{R} \). Since \( \lim_{\sigma \to \infty} L(s) - R(s) \not= 0 \), there exists \( \sigma_0 > 0 \) such that \( L(s) - R(s) \not= 0 \) for \( \Re s > \sigma > \sigma_0 \). It follows that \( \beta = \sigma \) for all real part of zeros of the function \( L(s) - R(s) \). We set \( R(z) = P(z)/Q(z) \) where the degrees of \( P, Q \) are \( p, q \), respectively; and define
\[
\tilde{\ell}(s) = (s) - R(s).
\]
Thus, there is \( \eta > 1 \) such that \( \tilde{\ell} \) is analytic in the region \( |s| > \eta \) since \( L \) is a meromorphic function in \( \mathbb{C} \) with the only pole at \( s = 1 \). We apply Littlewood’s argument principle [3] to \( \tilde{\ell} \) in the rectangle \( R = \{ \sigma + it : b \leq \sigma \leq c, T \leq t \leq 2T \} \) where \( c, T \) are parameters satisfying \( c > \max \{ \sigma, 1/b \}, T > \eta \). Thus,
\[
\int_{\sigma}^{\sigma+2\pi} \log |\tilde{\ell}(s)| \, ds = -2\pi \int_{b}^{\infty} \nu(\sigma, R) \, d\sigma
\]
where the given logarithm is defined as in Littlewood’s argument principle [3]. To prove our result, however, we first decompose our auxiliary function by

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\[ \tilde{\ell}(s) = \begin{cases} P(s) \left( \frac{L(s)}{P(s)} - 1 \right) & \text{for } p \leq q \\ R(s) \left( \frac{L(s)}{R(s)} - 1 \right) & \text{for } p > q \end{cases} \] (1)

Without loss of generality, we may assume that \( p, q \geq 1 \) whenever \( p \leq q \) since we can always write \( \tilde{\ell}(s) \) that exhibit polynomial growth, which is necessary for our proof. In the case of \( p > q \), \( R \) already exhibits polynomial growth, and no such adjustment is necessary. We now integrate the logarithm of \( \tilde{\ell} \) to get

\[
\int_{\mathcal{R}} \log \tilde{\ell}(s) ds = \begin{cases} \int_{\mathcal{R}} \log \ell_1(s) ds + \log P(s) ds + O(T) & \text{for } p \leq q \\ \int_{\mathcal{R}} \log \ell_2(s) ds + \log R(s) ds + O(T) & \text{for } p > q \end{cases}
\]

where the \( O(T) \) terms are the integrals of the maximum contribution from writing \( \tilde{\ell}(s) \) as a sum of logarithms. By our choice of \( T \), both \( \log P \) and \( \log R \) are analytic in \( \mathcal{R} \). Hence, Cauchy’s Theorem gives

\[
\int_{\mathcal{R}} \log \tilde{\ell}(s) ds = \begin{cases} \int_{\mathcal{R}} \log \ell_1(s) ds + O(T) & \text{for } p \leq q \\ \int_{\mathcal{R}} \log \ell_2(s) ds + O(T) & \text{for } p > q \end{cases}
\] (2)

To connect this integral with Littlewood’s argument principle [3], we note that the definition of \( c \) guarantees that

\[
-2\pi i \sum_{\beta_k \neq 0} (\beta_k - b) = i \text{Im} \left[ \int_{c-iT}^{c+iT} \log \ell_k(\sigma) d\sigma + i \int_{c-iT}^{c+iT} \log \ell_k(c+it) dt \right] + O(T)
\]

\[
= -i \int_{c-iT}^{c+iT} \log \ell_k(b+it) dt - \int_{c-iT}^{c+iT} \log \ell_k(c+it) dt - \int_{c-iT}^{c+iT} \arg \ell_k(\sigma + iT) d\sigma + \int_{c-iT}^{c+iT} \arg \ell_k(\sigma + iT) d\sigma + O(T)
\]

\[
= \sum_{j=1}^{\tilde{M}} I_{j,k} + O(T),
\]

for instance.

We now estimate \( I_{1,k} \). For \( T \) large enough, we have

\[
\log \left| \ell_1(b+it) \right| \leq \log \left| \frac{L(b+it)}{P(b+it)} - 1 \right| \leq \log \left( \frac{L(b+it)}{P(b+it)} \right) + \log \left( \frac{1}{Q(b+it)} \right)
\]

\[
\leq \log \left( |L(b+it)| + 1 \right) = \log \left( |L(b+it)| + 1 \right) \leq \log \left( |L(b+it)| + 1 \right) + \log 2.
\]

Then for \( T \) large enough, \( t \geq T, k = 2 \), we find in a similar fashion that

\[
\log \left| \ell_2(b+it) \right| \leq \log \left( |L(b+it)| + 1 \right) + \log 2.
\]

Since we have the same estimate for \( k = 1,2 \), we find that

\[
I_{1,k}(T,b) = I_{1,k} \leq T \int_{c-iT}^{c+iT} \log \left| b+it \right| dt + O(T)
\]

\[
= T \int_{c-iT}^{c+iT} \left( \frac{1}{T} + \frac{1}{T} \log |L(b+it)| dt \right) + O(T)
\]

where the final bound follows from Jensen’s inequality.
It is known [2] that for \( b > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\} \),
\[
\lim_{T \to \infty} \frac{1}{T} \left[ L(b + it) \right]^2 \, dt = \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^{2\sigma}} = O(1).
\]
Hence, \( I_{1,k}(T,b) \leq O(T) \) uniformly in \( b > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\} \).

We next move to estimate \( I_{2,k} \). For sufficiently large positive real number \( c \), we have
\[
\left| \frac{L(c+it)}{P(c+it)} \right| \leq 1 \text{ and } \left| \frac{L(c+it)}{R(c+it)} \right| \leq 1,
\]
so
\[
\log |\ell_1(c+it)| \leq \log \left| 1 - \frac{L(c+it)}{P(c+it)} \right|
\]
since \( q \geq 1 \). Furthermore,
\[
\log |\ell_2(c+it)| = \log \left| 1 - \frac{L(c+it)}{R(c+it)} \right|
\]

Since we may take \( c \) large enough so that \( |\ell_1(c+it)| \leq 1 \), we may write \( \log \ell_1(c+it) \) using a Taylor series expansion in the rectangle \( \mathcal{R} \). For \( k = 1 \), we have after taking real parts that
\[
\log |\ell_1(c+it)| \leq \Re \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \right\} \leq \log \left| 1 - \frac{L(c+it)}{P(c+it)} \right|
\]

We now observe that for sufficiently large \( T \) and some constant \( M \) we have
\[
\int_{\mathcal{R}} \left| \frac{L(c+it)}{P(c+it)} \right|^2 (n_1 \cdots n_k)^s \, dt \leq \frac{T}{|P(c+iT)|^2} \leq M^{1-k} \leq 1,
\]
for \( k \in \mathbb{N} \) and
\[
\limsup_{k \to \infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} \right)^k = \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} < 1
\]

for sufficiently large \( c \). In light of these bounds and the definition of \( \mathcal{M} \), we have (6)
\[
|I_{2,k}| = -\Re \left\{ \sum_{k=1}^{\infty} \sum_{n_1}^{\infty} \cdots \sum_{n_k}^{\infty} \frac{a(n_1) \cdots a(n_k)}{(n_1 \cdots n_k)^s} \right\} \leq \sum_{k=1}^{\infty} \sum_{n_1}^{\infty} \cdots \sum_{n_k}^{\infty} \frac{a(n_1) \cdots a(n_k)}{(n_1 \cdots n_k)^s} \leq \sum_{k=1}^{\infty} \sum_{n_1}^{\infty} \cdots \sum_{n_k}^{\infty} \frac{1}{n^{2\sigma}} = O(1),
\]
By (5), \( \log|g_z(c)| \) is bounded. Further, it is clear from a property of \( L \) functions that we have

\[
|L(s)| \leq A|t|^\alpha, \quad t \to \infty, \quad s \to \infty;
\]

for some positive absolute numbers \( A, B \) in any vertical strip of bounded width. The same estimate must hold for \( g_z(z) \) as well. Thus, the integral in (8) is \( O(\log T) \), implying that \( \hat{\nu}, \hat{\lambda}_z(R) = O(\log T) \). Since the interval \( [b, c] \subseteq D(c, R) \), it follows that

\[
N \leq \hat{\nu}, \hat{\lambda}_z(R') = O(\log T).
\]

With this bound, we integrate (7) to deduce that

\[
|I_{j,k}| \leq \int_{\gamma} |\arg \ell_k (\sigma + it)|d\sigma \leq \int_{\gamma} (N + 1)\pi d\sigma = O(\log T).
\]

As previously noted, we may bound \( I_{j,k} \) in the same way. Thus, we attain the desired bounds for \( j = 1, \ldots, 4 \) and \( k = 1, 2 \). Consequently, the first part of the theorem is proved by using (4).

**Proof of (II).** As in the proof of the first part of the theorem, we conclude that there exists a real number \( \sigma_0 \) for which the real parts \( \beta_k \) of all \( R \)-values satisfy \( \beta_k < \sigma_0 \); and also, there exist \( B, T' > 0 \) for each rational function \( R \) such that no zeros of

\[
L(s) - R(s) = 0 \quad \text{in the quarter-plane} \quad \sigma < -B, t > T'.
\]

As before, we define the rectangle \( R = \{ s = \sigma + it : b \leq \sigma \leq c, T \leq t \leq 2T \} \) where \( b, c, T \) are parameters satisfying \( b < -B - 1, c > \max \{ \sigma_0 + 1, b \}, T > \max \{ \sigma_0 + 1, b \} \).

Proceeding as in the proof of the first part of the theorem, we see that

\[
\begin{align*}
2\pi i \sum_{r \leq y \leq 2T} (\beta_k - b) &= -i \int_{-2T}^{2T} \log |\ell_k (b + it)| dt - \int_{-2T}^{2T} \log |\ell_k (c + it)| dt \\
&= \int_{y} \arg \ell_k (\sigma + i) d\sigma + \int_{y} \arg \ell_k (\sigma + i2T) d\sigma + O(T) \\
&= I_1 + \sum_{j=2}^{4} I_{j,k} + O(T)
\end{align*}
\]

for \( k = 1, 2 \) where \( \ell_k \) is defined as in (1). In the equation above, we note that we have chosen to compute \( I_1 \) separately. Indeed, this is the only estimate that we will need. For the integrals \( I_{j,k}, j = 2, 3, 4 \) and \( k = 1, 2 \), the bounds given as in the proof of the first part of the theorem still hold. First, integral \( I_{2,k} \) is unchanged. On the other hand, the integrals \( I_{3,k}, I_{4,k} \) have been altered by our choice of \( B \), but, as we have done as before, we still have the desired bound since the only requirement is that we consider \( L \) in a vertical strip of fixed width, which we have in this case.

We now bound \( I_1 \). Since \( b < -B \), we have by the functional equation in the definition of \( L \) function,
We now consider the last term in (9). Since,
\[ \limsup_{t \to \pm \infty} \frac{\log |L(b+it)|}{\log t} = \left( \frac{1}{2} - b \right) d_L, \]
and noting \( b < 0 \), we have for any \( \delta > 0 \) and \( t \geq T \)
\[ |L(b+it)| \geq \left| \left( \frac{1}{2} - b \right) d_L - \delta \right| \]
for sufficiently large \( T \). Then we see the quotient
\[ \left| \frac{R(b+it)}{L(b+it)} \right| \leq \frac{R(b+it)}{\left| \left( \frac{1}{2} - b \right) d_L - \delta \right|} = O\left( \frac{1}{t} \right) \]
when \( -b \) is large enough so that
\[ \deg R < \left( \frac{1}{2} - b \right) d_L - \delta + 1. \]
Therefore, we find that
\[ \log \left| 1 - \frac{R(s)}{L(s)} \right| = O\left( \frac{1}{t} \right). \]
Integrating in light of these estimates, we see
\[
\int_T^{2T} \log |L(b+it) - R(b+it)| \, dt \\
= \left( \frac{1}{2} - b \right) \int_T^{2T} (d_L \log t + \log(\lambda Q^2)) \, dt \\
+ \int_T^{2T} \log |L(1-b-it)| \, dt + O(\log T).
\]
The first integral is \( d_L T \log \frac{4T}{e} + T \log(\lambda Q^2) \), and the second integral is \( O(1) \) for sufficiently large and negative \( b \) by the method used to derive (6). Hence,
\[ I_i = \left( \frac{1}{2} - b \right) \left( d_L T \log \frac{4T}{e} + T \log(\lambda Q^2) \right) + O(\log T). \]

With the estimates for the \( I_{j,k} \)'s, we have proved the second part of the theorem.

REFERENCES