ABSTRACT

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Title: Reflective Abstraction and the Concept of Limit: A Quasi-Experimental Study to Improve Student Performance in College Calculus by Promoting Reflective Abstraction through Individual, Peer, Instructor and Curriculum Initiates

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ABSTRACT

This study is designed to improve student performance on the concept of limit by promoting reflective abstraction through instructor, peer, curriculum and individual initiates. It is based on Piaget's notion of reflective abstraction. It examines Piaget's four constructs of reflective abstraction, which are interiorization, coordination, encapsulation, and generalization. In addition it includes the notion of reversal, which is originally discussed by Piaget and refined into a construct of reflective abstraction by Dubinsky.

This study examined the performance of two sections of first-semester calculus students at a midwestern community college. One section of students studied an experimental curriculum designed to promote evidence that implies reflective abstraction occurs through the five constructs. These students completed problems in collaborative groups. They were required to identify the connections among the various topics and they were given several opportunities to reflect on and write about their understanding of the concepts. A second section of students studied a traditional curriculum. Students in both sections examined the same examples and completed the same homework exercises. Data for the study included pretest scores and posttest scores for all students in the study. Data also included

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transcribed interviews, homework sets, and classroom observations for a subgroup of 12 students.

The pretest-posttest scores showed that the students in the reflective abstraction section scored significantly higher than the students in the traditional section on a posttest of limits. An examination of the subgroup of students showed that the students in the reflective abstraction section scored significantly higher than the students in the traditional section on a measure of written communication. Further analysis of the data shows that successful students, regardless of assigned section, engaged in activities that imply reflective abstraction more often than the less successful students.
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REFLECTIVE ABSTRACTION AND THE CONCEPT OF LIMIT: A QUASI-EXPERIMENTAL STUDY TO IMPROVE STUDENT PERFORMANCE IN COLLEGE CALCULUS BY PROMOTING REFLECTIVE ABSTRACTION THROUGH INDIVIDUAL, PEER, INSTRUCTOR AND CURRICULUM INITIATES

A DISSERTATION SUBMITTED TO THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICAL SCIENCES

BY

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DEKALB, ILLINOIS

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DEDICATION

To Kathleen
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CHAPTER 1
INTRODUCTION

Calculus plays a vital role in the undergraduate curriculum. One of the key concepts in calculus is the limit. It is the first calculus topic which students encounter that is substantially different from algebra (Cornu, 1991). Research indicates that many students struggle with this concept (Tall, 1992).

In order to improve students' conceptual understanding of the concept of limit, a theoretical framework is needed to examine how and why students learn certain mathematical concepts. Reflective abstraction, as defined by Piaget (Beth & Piaget, 1966) and refined by Dubinsky (1991), is such a framework.

Theoretical Framework

Piaget (Beth & Piaget, 1966) describes reflective abstraction with its processes of assimilation and accommodation as a key element in developing conceptual understanding. Piaget examines four categories of reflective abstraction. These categories are interiorization, coordination, encapsulation and generalization. Dubinsky (1991) refines Piaget's notion of reversibility into a fifth category of reflective abstraction called reversal. This study is designed to address the difficulties students have with learning the concept of limit. These difficulties have
been established by Cornu (1991), Cottrill et al. (1996), Simonsen (1995), Norman and Prichard (1994), and Tall and Vinner (1981), among others. Cobb, Boufi, McClain and Whitenack (1997) claim that the role of the teacher is to initiate changes in the students in order to promote individual reflection. The term “initiate” is borrowed from Cobb et al. (1997) so that one can ask if it is possible to improve student performance in mathematics by initiating reflective abstraction.

This study examines a curriculum designed to initiate reflective abstraction on the concept of limit. Specifically, this curriculum includes initiates from the individual, peers, instructor, and curriculum. Two fundamental questions are posed:

1. Does the evidence imply that reflective abstraction occurs?
2. If so, does it improve student performance?

**Conceptual Understanding**

Why do some students successfully learn calculus and others do not? In order to answer these questions, one must examine the pivotal role of conceptual understanding in mathematics. “The need to understand and be able to use mathematics in everyday life and in the workplace has never been greater” (National Council of Teachers of Mathematics, 2000, p. 4).

with understanding, students should be able to apply their knowledge to new, unfamiliar situations. Bransford, Brown and Cooking (1999) conclude that conceptual understanding is an important component of proficiency. In order to examine conceptual understanding, one must attempt to define it.

Conceptual understanding can be viewed as the development of an interconnected network of abstract cognitive structures (Hiebert & Carpenter, 1992). Schoenfeld (1988) recognizes that mathematics will make more sense, be easier to remember, and be easier to apply if students are able to meaningfully construct new knowledge and connect that knowledge to existing structures. However, conceptual understanding is a challenge for many students. Davis and Vinner (1986) argue that the development of abstract mental concepts is an unusual activity that many students find difficult. They believe that students tend to avoid concept development. Instead these students focus on notation and symbols with little regard for meaning.

The essential question is, “How can students develop conceptual understanding?” Piaget helps answer this question with his discussion of reflective abstraction.

Reflective Abstraction

Piaget (Beth & Piaget, 1966) states that students construct knowledge through the process of reflective abstraction. The fundamental components of reflective abstraction are assimilation and accommodation. Assimilation is the active process of constructing a new cognitive structure and accommodation is the
active process of revising that structure so that it fits coherently with existing structures. In describing Piaget’s work, Noddings (1990) writes, “This active construction implies both a base structure from which to begin construction (a structure of assimilation) and a process of transformation and creation which is the construction. It implies, also, a process of continual revision of structure (a process of accommodation)” (p. 9).

According to Piaget, reflective abstraction is present at the earliest stages of cognitive development, and this process continues throughout advanced mathematics (Beth & Piaget, 1966). In fact, the development of modern mathematics from primitive mathematics can be viewed as a process of reflective abstraction (Piaget, 1985). The mathematics education community has further refined Piaget’s concept of reflective abstraction into modern definitions.

Definitions of Reflective Abstraction

Piaget focuses on mathematical development of young children. Dubinsky (1991) claims that Piaget’s model is easily adapted to advanced mathematics. In order to examine the cognitive development necessary in learning advanced mathematics, Dubinsky and his students formed the Research in Undergraduate Mathematics Education Community (RUMEC). This group examines the work of Piaget and Garcia (1983) in *Psychogenèse et histoire des sciences* in order to define reflective abstraction as follows:

Reflective abstraction is a concept introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. Reflective abstraction by an individual
proceeds from two mechanisms which are necessarily associated. They are projection unto a higher level of that which was derived from a lower level, and secondly reflection, which reconstructs and reorganizes within a larger system that is transferred by projection. (DeVries, 2001)

Cooley (2002) clarifies the notion of reflective in the following passage.

Reflective abstraction is a mechanism for the isolation of particular attributes of a mathematical structure that allows the subject to construct or reconstruct knowledge that is new, that is, knowledge not previously known. A feature of reflective abstraction is that it clarifies and organizes logico-mathematical experiences in such a way as to recognize both nuances and broad generalizations among them. Any new constructions will be associated with knowledge the subject already has. The subject orders or re-orders a class of situations with the characteristics of the current object so that the new knowledge fits with previous schemas, or the previous schema has been reconstructed. The new generalization occurs precisely because of a mental construction or reconstruction. (Cooley, 2002 p. 255)

Many authors discuss student difficulties in learning concepts. For example, Sierpinska (1987) discusses epistemological obstacles and Vinner and Dreyfus (1989) discuss difficulties between concept image and concept definition. These frameworks are very important, but reflective abstraction has a distinct advantage. Reflective abstraction explains how cognitive structures are developed rather than why they are not developed (Dubinsky, 1991).

** Constructs of Reflective Abstraction **

Reflective abstraction is a thought process that occurs within an individual. It is impossible to know exactly what happens in the mind of another individual. Therefore reflective abstraction must be inferred based on evidence. What are the aspects of reflective abstraction that can be inferred? Piaget (Beth & Piaget, 1966) describes four constructs of reflective abstraction. These are interiorization,
coordination, encapsulation, and generalization. Piaget describes the importance of reversal in cognitive development (Piaget, Inhelder & Szeminska, 1960). Dubinsky (1991) refines the concept of reversal into a fifth construct and he claims that this is essential in advanced mathematics. Descriptions of the five construct follow.

**Interiorization**

Piaget defines interiorization as “translating a succession of material actions into a system of interiorized operations” (Beth & Piaget, 1966, p. 206). Dubinsky (1991) describes interiorization as the construction of internal processes in order to make sense of mathematical concepts. The tools used in constructing these processes include symbols, pictures and language. An example of interiorization is a child seeing the symbols 2 + 3 and recognizing that she must start with a set of two objects and she must imagine another set of three objects. From these mental images she constructs a new set that includes all of the original elements. She counts all of the elements. She has internalized the process of “addition” to be counting all of the objects after two sets are joined.

**Coordination**

This construction is the process of coordinating two or more processes to obtain a new process (Dubinsky, 1991). An example of coordination can be seen with 12 + 29. Rather than simply constructing two sets, joining, and counting, the child may decompose the problem as follows: 12 = 10 + 1 and 29 = 30 – 1. She may coordinate following processes of (a) decomposition, (b) commutativity of
addition, (c) count all strategy for addition, (d) count up strategy for addition and (e) count down strategy for subtraction.

\[ 12 + 29 = 10 + 2 + 30 - 1 \]  
\[ \text{Decomposition} \]

\[ 10 + 30 + 2 - 1 \]  
\[ \text{Commutativity} \]

\[ 40 + 2 - 1 \]  
\[ \text{Count all} \]

\[ 42 - 1 \]  
\[ \text{Count up} \]

\[ 41 \]  
\[ \text{Count down} \]

Encapsulation

Dubinsky and Lewin (1986) state, “Perhaps the most important form of reflective abstraction involves a process of encapsulation” (p. 62). Dubinsky (1991) defines encapsulation as the conversion of a dynamic process into a static process. Piaget (1985) writes, “Actions or operations become thematized objects of thought or assimilation. . . . The whole of mathematics may therefore be thought of in terms of construction of structures, . . . mathematical entities move from one level to another; an operation on such entities becomes in its term an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by stronger structures” (p. 49). An example of encapsulation is a child understanding that \(3 + 3 + 3 + 3 + 3 + 3 + 3\) can be expressed as seven sets of three, thus encapsulating the notion of product from the process of repeated addition. Another example is a student being told to think of a number, double it and then add one. If this student determines that this process is the function
\[ f(x) = 2x + 1, \] he has encapsulated the notion of function from the original process. Now the student could use this function to add to other functions or compose with other functions. In a sense the process has become an object.

**Generalization**

Generalization occurs when a student applies an existing schema to a wider collection of concepts. Dubinsky and Lewin (1986) describe the relationship between generalization and encapsulation in the following: "A structure is, in some sense, a form, acting on various aliments as content. After encapsulation this form can become content for other structures which, when generalized, can act upon the encapsulated structure as an aliment" (p. 63). An example of encapsulation is a student using an encapsulated notion to solve an application problem. For example, a student may have encapsulated addition of integers. If she sees the problem, "Mary has $20. She owes Bruce $12. She owes Carla $15 and she gets a $10 gift from her grandma. After paying her debts, how much does Mary have," the student may rewrite the question as \[ 20 + (-12) + (-15) + 10 = 3, \] thus generalizing the notion of addition of integers to a financial problem.

**Reversal**

Piaget does not include reversal as one of his constructs of reflective abstraction, yet he discusses its importance in concept development (Piaget, Inhelder & Szeminska, 1960). Dubinsky (1991) refines Piaget's notion of reversal into a construct of reflective abstraction. Dubinsky defines reversal as constructing...
a new structure through the process of reversing the original structure. For example, a student who knows \(2 + 3 = 5\) can reverse the process to conclude that \(5 - 2 = 3\) and \(5 - 3 = 2\).

Reflective abstraction is a promising framework for examining conceptual understanding in mathematics and since it can be extended to advanced mathematics it is well suited for examining calculus. Since many students have difficulty learning calculus, research of this type is relevant.

**Initiates of Reflective Abstraction**

Recognizing that reflective abstraction is an individual activity, Cobb et al. (1997) claim that the teacher is capable of initiating shifts in the discussion that may lead to reflection. Hershkowitz and Schwarz (1999) state that a rich learning experience promotes reflective processes. Therefore the research indicates that a teacher and a curriculum are capable of initiating reflective abstraction, which in turn promotes conceptual understanding.

Borrowing the term “initiate” as used by Cobb et al. (1997), let us examine how to “initiate” reflective abstraction. First, Piaget (Beth & Piaget, 1966) claims that reflective abstraction is a personal activity; therefore, the individual student is capable of initiating reflective abstraction. Second, Cobb, Jaworski, and Presmeg (1996) discuss the relationship between social discourse and individual reflective abstraction. “Classroom discourse and social interaction can be used to promote the recognition of connections among ideas and reorganization of knowledge” (National Council of Teachers of Mathematics, 2000, p. 21). Therefore peers are
capable of initiating reflective abstraction. Third, Cobb et al. (1997) claim that the teacher is capable of initiating shifts in the discussion which may lead to reflective abstraction, so the teacher is capable of initiating reflective abstraction. Finally, several curricula have been developed in recent years to encourage students to begin to reflect about their thinking in mathematics. These include the Harvard Project (Hughes-Hallett, 1997), Project CALC (Smith & Moore, 1991), Calculus and Mathematica (Davis, Porta & Uhl, 1994) among others. So it appears that a curriculum may initiate reflective abstraction.

Teachers, peers and the curriculum may initiate reflective abstraction but they cannot guarantee it. Piaget (Beth & Piaget, 1966) claims that reflective abstraction is an individual activity. Despite the best efforts by teachers and curriculum designers, the individual alone is capable of engaging in reflective abstraction, but the teacher may infer reflective abstraction based on student performance.

State of Calculus

Calculus plays an essential role in the college mathematics curriculum. Engineering, science, and mathematics students must complete calculus before studying upper-level quantitative courses. The fundamental role of calculus is seen in the following statement: “Almost all of science is concerned with the study of systems of change, and the study of change is the very heart of the differential calculus. . . . Thus all science and engineering students need calculus in their
studies" (Douglas, 1986, p. iv). However, science and engineering majors are not the only students enrolled in calculus. In addition, architecture, computer science, pre-medical, and business students conclude their traditional mathematics studies with calculus.

Problems with Calculus

Despite its importance, calculus is in crisis. Dropout and failure rates in calculus are excessively high. Even those who pass perform poorly on calculus skills and concepts (Cipra, 1988; Peterson, 1986). Selden, Selden and Mason (1994) demonstrate that the best calculus students are unable to successfully complete nonroutine calculus problems. Epp (1987) writes, "The fact is that the state of most students’ conceptual knowledge of mathematics after they have taken a calculus course is abysmal" (p. 48).

This state is shared throughout university mathematics programs. The Committee on the Undergraduate Program in Mathematics (2004) claims that the total number of mathematics majors is decreasing and the enrollment in advanced mathematics classes is also declining. Clearly the crisis in calculus is a significant factor in this problem.

Calculus Reform

In an attempt to improve calculus teaching and learning, the Mathematics Association of America published two documents, *Toward a Lean and Lively Calculus* (Douglas, 1986) and *Calculus for a New Century* (Steen, 1987). The
authors of these documents recommended active student participation in the calculus. They suggest that students should do calculus through a laboratory model where students engage in nonroutine and open-ended problem solving. The authors also encourage instructors to stress conceptual understanding. The changing state of calculus requires investigation. Douglas (1986) writes,

> We need to know more about what students learn in their mathematics classes. A close look at students’ work (by means of interviews, videotapes of students working problems, etc.) is often a disturbing, but valuable source of information. More detailed research on students’ mathematics learning would be helpful, both to tell us about current difficulties in instruction and to suggest ways that might help us to improve. (p. 3)

Schoenfeld (1997) asks for an effective strategy to assess student understanding in calculus and he feels that the newly designed curricula must be examined to determine whether or not they are effective in helping students learn calculus.

Since calculus is a vast area of mathematics, and since the concept of limit is one of the earliest and most important concepts in the curriculum, it is a natural topic to study.

**Concept of Limit**

Tall (1992) claims that “although the function concept is central to modern mathematics, it is the concept of a limit that signifies a move to a higher plane of mathematical thinking” (p. 501). Cornu (1991) believes that the limit is typically the first mathematical concept that students encounter where one cannot get a
solution with a direct computation. He claims that limits are "surrounded with mystery . . . one must arrive at one's destination by a circuitous route" (p. 151).

Sierpinska (1987) describes the misconception that many students have as a "static" perspective of limit. Tall (1992) discusses the misconceptions from everyday language that cloud much of the meaning of limit. He notes that a "speed limit" should not be exceeded. He mentions "approaches" and "gets close to" mean different things in common language versus mathematical language. Davis and Vinner (1986) discuss how the examples that students study create an image that is not wholly accurate. For instance, most examples done in class are monotonic functions, so students come to believe that for a limit to exist, the function must be monotonic.

Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic (1996) discuss the difficulties that students have with the formal concept of limit. The authors recognize that many students are unable to coordinate the necessary processes to understand the concept. Sierpinska (1987) and Tall and Vinner (1981) recognize that the formal definition requires knowledge of quantification that is beyond many students. This leads to the question of examining reflective abstraction in order to study how students come to understand the concept of the limit.

Question

Can a curriculum that initiates reflective abstraction improve student performance on the concept of limit?
Piaget (1972) claims that reflective abstraction is an individual activity. Cobb, Jaworski and Presmeg (1996) discuss the role of discourse in promoting reflective abstraction. They claim that reflective discourse creates the conditions for mathematical learning but such learning is not inevitable. Cobb et al. (1997) claim that discourse does not cause reflection. Rather, the individual student must reflect and reorganize her own cognitive structures. This research project is designed to determine if a curriculum can improve student performance by initiating reflective abstraction.

Overview of Research Design

An experimental design is used. One section of calculus students studied a control curriculum. These students studied limits in a traditional manner. Protocols that describe the students' understandings of limits were collected. These protocols were analyzed for spontaneous occurrences of the five constructs of reflective abstraction.

A second section of calculus students studied an experimental curriculum. This curriculum is designed to initiate reflective abstraction. These initiates may come from the individual, peers, the instructor or the curriculum itself. Again, protocols that describe the students’ understandings of limits were collected. These protocols were also analyzed for the five constructs of reflective abstraction.

The qualitative component of the study includes a comparison of the performance of the two groups. It documents inferences of reflective abstraction. This component describes to what extent the experimental curriculum initiates
reflective abstraction as well as the extent the control curriculum initiates reflective abstraction.

The quantitative component of the research compares the control group and experimental group with respect to performance on a test of limits. The quantitative component explains which group did better and the qualitative component helps to explain why.

Conclusion

Calculus is a gateway into many technological and scientific fields, yet it is an impediment for many students. The topic of limit is the first sophisticated concept one studies in calculus. Reflective abstraction together with its constructs of interiorization, coordination, generalization, encapsulation and reversal is a promising area of research. For these reasons, a study that initiates reflective abstraction in order to improve student performance on the concept of limit is valuable.
CHAPTER 2
A REVIEW OF THE LITERATURE

Genetic Epistemology

Piaget (1967) uses the term genetic epistemology to describe the development of knowledge, in particular scientific knowledge. He claims that knowledge is developed based on the history, the social interaction, and the psychological origins of the ideas upon which the knowledge is based. He is most concerned with transformations from one level of thought to another.

In order to describe genetic epistemology, Piaget (1967) examines the historical development of science. He argues against the notion that knowledge is static. He claims that knowledge is an ever-developing process. New ideas from science require a continual construction and reorganization of ideas.

Piaget (1967) also believes that more abstract notions are often based on elementary concepts. For example, Piaget discusses Cantor set theory and he recognizes that this concept comes from the elementary notion of one-to-one correspondence.

Piaget (1967) criticizes the logical positivists who deny the role of psychology in the development of knowledge. Piaget claims that the positivists equate mathematics with linguistic structures. He believes that there is much more to mathematical concepts than simply rules of grammar and syntax. Piaget also
criticizes rationalists who believe that language is derived from logic. Piaget argues that there is no one logic and he points to the inadequacy of logic when he refers to Gödel's theorem that no rich axiomatic system can prove its own consistency. Piaget argues that the development of knowledge requires not only logic but also psychology.

Piaget (1967) criticizes the view that a student's knowledge is simply a copy of the teacher's knowledge. Piaget writes, "To my way of thinking, knowing an object does not mean copying it – it means acting upon it. It means constructing systems of transformations that can be carried out on or with this object" (p. 9).

Piaget (1967) claims that knowledge is derived from history, social interaction, and psychological origins of underlying concepts. He criticizes positivist and rationalist epistemologies as being insufficient for the development of scientific knowledge. How does Piaget claim that knowledge is developed? It is developed through the process of reflective abstraction.

Reflective Abstraction

Piaget introduces the term reflective abstraction in order to describe how an individual constructs mathematical structures. He claims that reflective abstraction occurs during the earliest stages of cognitive development (Beth & Piaget, 1966) and that this process continues to occur in the development of advanced mathematical structures. In fact, Piaget claims that the development of modern mathematics from ancient times to the present can be viewed as a process of reflective abstraction (Piaget, 1985).
Piaget (2001) discusses reflective abstraction as a means of criticizing previously mentioned theories of mathematical development. He believes that the development of sophisticated mathematical concepts requires abstraction of ideas. He writes,

But to draw an operation of a higher-level character (operatory seriation does not appear in children till around 7 years of age) out of a conduct at a somewhat lower level (evocative memory undoubtedly begins with language), we must appeal to a unique kind of abstraction. This is precisely abstraction from actions as opposed to abstraction from objects. (p. 9)

Piaget (2001) argues that intuitions are not easily abstracted, nor are they automatically incorporated into higher-level forms. In contrast, he claims that abstractions of ideas can become part of a larger structure.

Now in opposition to this abstraction of physical qualities, the abstraction of a mental characteristic that qualifies some action scheme and is destined to bring this characteristic into a more complex scheme (not just into a simple descriptive concept of internal experience) is reflecting experience. Calling it reflecting indicates that abstraction transforms the very conduct by differentiating it and consequently adds something to the quality that has been isolated by abstraction. (p. 10)

In comparing reflective abstraction to physical abstraction, Piaget (2001) writes,

Thus reflecting abstraction is essentially differentiation. It leads to a generalization that is a novel composition, preoperatory or operatory, because it involves a new scheme that has been elaborated by means of elements borrowed from prior schemes by differentiation. And the new scheme is more mobile and more reversible, and consequently more equilibrated. (p. 11)

Henning (1998) summarizes Piaget’s theories of reflective abstraction in the following:

(1) Wholeness (firstness) is disrupted by disequilibria (secondness) which motivates the organism to seek equilibration through assimilation and
accommodation (thirdness). (2) Operations or relationships become the basis for operations on the next level. (3) Equilibration is infinite; there is no final stopping point. (4) The learner’s understanding is determined by the previous schemas she has constructed. (p. 9)

Henning (1998) uses a furniture metaphor to describe Piaget’s notion of reflective abstraction. He looks at his neighbor’s furniture and this causes him to re-evaluate his previous notions of furniture. His neighbor’s furniture is new and expensive. His furniture is old and inexpensive. Therefore he has to eliminate age and cost as characteristics of furniture. He redefines furniture as usable items, made of wood, purchased in a furniture store. Over time this definition will have to change, as his furniture is no longer usable. Reflecting on shared characteristics and designing a classification scheme is an example of reflective abstraction.

Campbell, a translator of Piaget (2001), describes reflecting abstraction in the following: “It leads to constructive generalizations, to genuinely new knowledge, to knowledge at higher levels of development, and to knowledge about knowledge” (p. 12).

Reflecting abstraction was not seen as a prominent element of Piaget’s first edition, but the reissue recognized it as one of the key ideas of the work (Piaget, 2001). Campbell claims that Piaget’s preface to the second edition concluded with what Piaget thought were the major accomplishments. The following excerpt is from the preface:

Where Aristotelian abstraction abstracts from objects, reflecting abstraction draws its information from the subject’s actions on objects – which are not the same thing – and particularly from the coordination between these actions. Thus it provides to higher levels the reasons for the connections that have been extracted from lower levels. It is this fundamental process which seems to be the home for the continual creation of norms, which our
previous remarks indicated are the essential cognitive characteristics of the knowing subject's activities. (p. 13)

Campbell (Piaget, 2001) states that there are several unanswered questions about reflecting abstraction. Piaget wrote about these ideas at the end of his career and he did not have sufficient time to resolve all of the concerns. Campbell includes the following questions:

1. Is reflecting abstraction qualitatively the same as empirical abstraction?
2. How close is the connection between reflecting abstraction and equilibration?
3. What is the connection between reflecting abstraction and consciousness?
4. Can reflecting abstraction go wrong? Is it allowed to produce errors?

Despite these questions, reflective abstraction is considered to be an effective theoretical framework for examining student understanding in mathematics. Von Glaserfeld, Thompson, Cifarelli, and Dubinsky describe the role of reflective abstraction in learning mathematics.

Von Glaserfeld (1991) describes reflection as a process of re-presentation. He writes,

I know of no better description of the mysterious capability that allows us to step out of direct experience, to re-present a chunk of it and to look at it as though it were direct experience, while remaining aware of the fact that it is not... Focused attention picks a chunk of experience, isolates it from what came before and from what follows and treats it as a closed entity. For the mind, then, "to posit it as object against itself" is to re-present it. (p. 47)

The process of re-presentation allows a person to re-generate a prior experience. This regeneration can occur because the original experience leaves marks that enable the reconstruction.
Von Glaserfeld (1991) describes mathematical symbols as a type of marker that allows an individual to re-present a notion. He writes,

The word/symbol must be associated with a conceptual structure that was abstracted from experience and, at least to some extent, generalized. Here, again, the ability to recognize (i.e., to build up the conceptual structure from available perceptual material) precedes the ability to re-present the structure to oneself spontaneously. (p. 52)

He claims that symbols can help individuals isolate meaning. The symbol becomes a “pointer” that enables the student to re-present an idea at a later time. Symbols help students represent ideas. With a symbol, the ideas are more easily generalized into other domains. Von Glaserfeld writes,

Once symbols have been associated with the abstracted operative pattern, these symbols, thanks to their power of functioning as pointers, can be understood, without the actual production of the associated re-presentation – provided the user knows how to produce it when the numerical material is available. (p. 63)

This idea is very similar to Tall’s notion of procept. Tall claims that a procept consists of a process that produces a mathematical object and a symbol that represents the process or object (Gray & Tall, 1994). Tall recognizes that many students struggle with understanding precepts. He notes that some students focus solely on the procedures and they have difficulty in developing understanding. Yet others are able to easily switch between process and concept and thus they develop stronger cognitive structures.

In describing Piaget’s work, Von Glaserfeld (1991) describes two important types of reflective abstraction. The first type coordinates operations so that the notions can be projected onto another level. The second, like the first, coordinates ideas but also includes awareness of what has been abstracted. The first notion
could be referred to as projection and adjusted organization. The second is referred to as reflected thought. Piaget clarifies this in the following: “Finally, we call the result of reflective abstraction ‘reflected abstraction’, once it has become conscious, and we do this independently of its level” (Piaget et al., 1977, p. 303).

Thompson (1991) agrees with Von Glaserfeld’s idea that reflective abstraction, re-presentation and representation are essential elements in understanding, yet Thompson also stresses the importance of intuition. Thompson writes, “Intuitive thought, then, is the formation of un-controlled schemes which themselves function to control aspects of cognitive functioning. But these un-controlled schemes are themselves part of the organism’s cognitive functioning, and hence are something to be controlled” (p. 266).

Cifarelli (1988) defines six levels of reflective abstraction for problem solving. These strategies focus on Von Glaserfeld’s notion of re-presentation. In each case the student level can be inferred based on problem-solving performance strategies.

The first level is the instrumental level. A student at this level is completely unreflective. He engages in an activity without reflecting on the type of possible solutions. Solving an algebra problem mechanically without reflection is an example of a student performing at the instrumental level.

The second is the recognition level. A student at this level recognizes that a certain strategy learned previously may help to solve a given problem. She recognizes the type of problem and selects the appropriate strategy. She does so with little reflection.
The third is the *reflection on a perceptual expression on a representation level*. A student at this level can create figures, diagrams, or other representations that can be used as an aid in problem solving.

The fourth is the *reflection on a re-presentation level*. Students at this level can successfully use previously constructed procedures. They have internalized the procedures and they can then be extended into areas that were not previously studied. However, the student may not be aware that her results may contradict her notion of a certain concept.

The fifth is the *structural abstraction level*. Students at this level have internalized strategies used to solve previous problems. These students are able to re-present potential strategies and solutions mentally and they can predict results.

The sixth is the *structural awareness level*. Students at this level can solve problems without re-presentation of the solution strategy. The strategy is internalized as a structure that does not need to be re-presented.

However, Piaget does not focus on advanced mathematics. Dubinsky (1991) claims that Piaget’s notions applied to elementary mathematics can readily be applied to advanced mathematical concepts. In particular, he claims that the process of reflective abstraction is a key component in advanced mathematical thinking.
Types of Abstraction

Piaget defines three similar topics: *empirical abstraction, pseudo-empirical abstraction,* and *reflective abstraction.* Empirical abstraction occurs when an individual examines an external subject and internalizes some information. An example of this type of reasoning is determining properties of a class of object (Piaget & Garcia, 1983). In examining pseudo-empirical abstraction, Dubinsky (1991) writes, “Pseudo-empirical abstraction is intermediate between empirical and reflective abstraction and teases out properties that the actions of the subject have introduced into objects” (p. 97). Reflective abstraction, on the other hand, is completely internal and occurs through the “general coordination” of actions (Piaget, 1980). In order to clarify these ideas, Dubinsky (1991) explains,

Empirical and pseudo-empirical abstraction draws knowledge from objects by performing (or imagining) actions on them. Reflective abstraction interiorizes and coordinates these actions to form new actions and, ultimately new objects (which may no longer be physical but rather mathematical such as a function or group). Empirical abstraction then extracts data from these new objects through mental actions on them, and so on. (p. 98)

He also writes,

Reflective abstraction differs from empirical abstraction in that it deals with action as opposed to objects and it differs from pseudo-empirical abstraction in that it is concerned, not so much with the actions themselves, but with the interrelationships among actions. (p. 99)

Steffe (1991) explains that as models of reflective abstraction emerge, there will be more creative work in constructive learning theory. However, he writes, “... reflective abstraction must be operationally defined in particular contexts with respect to particular schemes before it has any clear meaning” (p. 42). A careful
definition of reflective abstraction and how it is used in advanced mathematics is needed.

Definitions of Reflective Abstraction

The Research in Undergraduate Mathematics Education Community (RUMEC) has borrowed from the work of Piaget (Beth & Piaget, 1966) and Dubinsky (1991) in formulating a definition of reflective abstraction:

Reflective abstraction is a concept introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. Reflective abstraction by an individual proceeds from two mechanisms, which are necessarily associated. They are projection unto a higher level of that which was derived from a lower level and secondly reflection, which reconstructs and reorganizes within a larger system that is transferred by projection. (DeVries, 2001)

Cooley (2002) defines reflective abstraction as follows:

Reflective abstraction is a mechanism for the isolation of particular attributes of a mathematical structure that allows the subject to construct or reconstruct knowledge that is new; that is, knowledge not previously known. A feature of reflective abstraction is that it clarifies and organizes logico-mathematical experiences in such a way as to recognize both nuances and broad generalizations among them. Any new constructions will be associated with knowledge the subject already has. The subject orders or re-orders a class of situations with the characteristics of the current object so that the new knowledge fits with previous schemas, or the previous schema has been reconstructed. The new generalization occurs precisely because of a mental construction or reconstruction. (p. 255)

The Five Constructs of Reflective Abstraction

Piaget includes the constructs of *interiorization, coordination, encapsulation* and *generalization* in his discussion of reflective abstraction. Dubinsky (1991)
refines Piaget's (1950) concept of reversibility into a fifth construct of reflective abstraction. Dubinsky claims that this is essential in advanced mathematics. Dubinsky refers to this construct as *reversal*.

The first construct is interiorization. Dubinsky (1991) defines interiorization as using the symbols and language of mathematics in order to develop internal processes that assist in developing understanding. Piaget defines the term as “translating a succession of material actions into a system of interiorized operations” (Beth & Piaget, 1966, p. 206). An example of interiorization is a child seeing the symbol 2 + 3 and recognizing that she must start with a set of two objects and then she must imagine a set with three objects and finally she must count all of the objects in both sets. A calculus example of interiorization is plugging values closer and closer to 1 into a function \( f(x) \) in order to approximate \( \lim_{x \to 1} f(x) \).

The second construct is coordination. It is the process of coordinating two or more processes to obtain a new process. An example of coordination is a child solving 12 + 29. Rather than simply constructing two sets and counting all, the child may decompose the problem as follows: 12 = 10 + 2 and 29 is 30 – 1. She may coordinate the strategies of (a) commutativity of addition, (b) count all strategy for addition, (c) count up strategy for addition, (d) count down strategy for subtraction: 12 + 29 = 10 + 2 + 30 – 1 = 10 + 30 + 2 – 1 (commutativity), = 40 + 2 – 1 (count all), = 42 – 1 (count up), = 41 (count down). An example from calculus would be coordinating the processes of plugging values closer and closer to 1 into a
function \( f(x) \) in order to approximate \( \lim_{x \to 1} f(x) \) together with examining the graph of \( f(x) \) near \( x = 1 \) in order to determine whether or not \( \lim_{x \to 1} f(x) \) exists.

The third construct is encapsulation. Dubinsky and Lewin (1986) state, "Perhaps the most important form of reflective abstraction involves a process of encapsulation" (p. 62). Dubinsky (1991) defines encapsulation as the conversion of a dynamic process into a static process. Piaget (1985) writes,

Actions or operations become thematized objects of thought or assimilation . . . The whole of mathematics may therefore be thought of in terms of the construction of structures . . . mathematical entities move from one level to another; an operation on such entities becomes in its term an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by stronger structures. (p. 49)

An example of encapsulation is solving \( 3 + 3 + 3 + 3 + 3 + 3 + 3 \). When the student uses the count up strategy to determine the sum, she is interiorizing a process. However if she recognizes that this repeated addition can be written as seven sets of three and that it can be written as \( 7 \times 3 \), she has encapsulated the notion of product from the process of repeated addition. The process of repeated addition is encapsulated into the object product. A calculus example of encapsulation is (a) a student constructing a definition of a limit at a point \( c \); (b) using tables, graphs and algebra to support that definition; and (c) recognizing that the limit of a function at a point \( c \) can be viewed as an object referred to as \( \lim_{x \to c} f(x) \).

The fourth construct is generalization. This occurs when a student applies an existing structure to a wider collection of concepts. An example of this construct is a student generalizing from the notion of a function of real numbers to a vector-valued function. Dubinsky and Lewin (1986) describe the relationship between
generalization and encapsulation. They claim that once a structure is encapsulated it can be used as content for other structures. This results in extending or generalizing the structure. For example, a student may have an encapsulated notion of addition of integers and use it to solve an application problem. Mary has saved $20 and she needs $32 to buy a math book. How much more does she need to save? Therefore the structure of addition can be generalized to solve a missing addend problem. A calculus example is using the recently encapsulated structure of limit as content for the structure of slope of a line in order to construct the notion of slope of the tangent line at point c.

The fifth construct is reversal. Piaget (1950) discusses the importance of reversibility, however, he does not include reversal as one of his constructs of reflective abstraction. Dubinsky (1991) argues that reversal is crucial to development of mathematical structures. He defines it as constructing a new structure through the process of reversing the original structure. For example, a student who knows $2 + 3 = 5$ can reverse the process to conclude that $5 - 3 = 2$ and $5 - 2 = 3$. Dubinsky (1991) argues that reversal is especially important in advanced mathematical thinking. For example, once a student has encapsulated the notion of limit, he should be able to reverse the process by constructing a continuous function whose limit as $x$ approaches 1 is 5.

Reflective abstraction is an essential tool for the development of mathematical structures. Its components of interiorization, coordination, encapsulation, generalization and reversal help clarify the concept. Reflective abstraction is a suitable framework for examining learning in mathematics. It is
especially promising for examining advanced mathematical thinking like that which occurs in calculus.

**APOS Theory**

Dubinsky and his students use the theories of reflective abstraction in the development of their Action, Process, Object, Schema theory (APOS); (Weller et al., 2003). This theory, based in the foundation of reflective abstraction, describes the mental constructions a student might make in the process of understanding mathematical concepts.

An action is the process of transforming objects to obtain other objects. This is an external activity that students perform from memory or perform by following step-by-step procedures. There is little depth of understanding at this stage.

When an individual repeats an action, he may reflect on that action and interiorize the action into a mental process that he has some control over. Weller et al. (2003) indicate that a process conception requires a student to describe a process or reverse a process without actually performing the steps. A student remains in the process conception if their understanding is limited to the procedural context.

A student has an object conception if he reflects on the process and realizes that he can create transformations for that process. This individual has encapsulated the process into an object. This object can now be used as a tool in problem solving.
A schema is created for a certain mathematical concept. It is a collection of actions, processes, objects, or other schema that are coherently structured into a framework that a student can use when solving problems.

Dubinsky (1991) suggests how this model should be used in designing instruction. These suggestions include the following:

1. Teachers should observe students as they try to learn a particular mathematical concept, and these teachers should identify the various conceptual structures or concept images that the students develop.

2. Teachers should use their personal knowledge of the mathematical concepts together with the APOS theory and the observations described in (1) to develop a genetic decomposition that represents one way a student may construct a cognitive structure.

3. Teachers should design instruction that enables the students to complete the steps in the genetic decomposition. The teacher must use activities that initiate the appropriate types of reflective abstraction.

4. This process must be repeated with the genetic decomposition continually being refined and improved.

Several other studies have followed these suggestions by examining student performance in calculus using reflective abstraction and APOS as the theoretical framework. These include the development of students' graphical understanding of the derivative (Asiala et al., 1996), the schema triad – a calculus example (Baker, Cooley & Trigueros, 1999), constructing a schema, the case of the chain rule (Clark, et al., 1997) and the genetic decomposition of the limit (Cottrill et al., 1996). These
studies suggest that calculus instruction that follows this model may improve student performance in calculus, and, more importantly, it may help students develop more mature conceptualizations of mathematical concepts (Weller et al., 2003). Other authors have examined the role of reflection and reflective abstraction in mathematical development.

Reflective Discourse

Cobb et al. (1997) examine reflective discourse and collective reflection. They see two consequences of reflective discourse. First, reflective discourse helps students construct mathematical concepts. Second, reflective discourse orients students to mathematical activity. They claim that reflective discourse helps students develop a mathematical disposition. However, the authors stress that participation in reflective discourse does not cause students to reflect. Rather, it is an individual activity that may or may not occur as part of a group dynamic.

Hershkowitz and Schwarz (1999) examine the role of reflective discourse in a classroom community. In particular, they examine how the individual reflective processes occur as part of group dynamics in a classroom community. In the process of solving a problem, the teacher requires students to report their group findings to the entire class. The teacher mediates these reports and helps the students to either appropriate or reject the findings of the group. The authors refer to this as the process of purification. They claim that the process of reporting promotes reflection and purification.
Cooley (2002) uses writing exercises to encourage reflective abstraction among her calculus students. Initially students showed little reflective abstraction in the writing. As the term progressed, student performance improved. Her students successfully classified and organized information. They discussed relationships among the various topics, and they began to make generalizations when appropriate. Cooley also recognizes that the student writing improved the instructor’s awareness of student understanding and student difficulties. However, Cooley is not certain if the writing assignments are the catalysts for reflective abstraction.

Calculus

Calculus plays a central role in the undergraduate mathematics curriculum. The goal of much K-12 mathematics instruction is to prepare students for calculus. Almost all science and engineering students must study calculus, and calculus is the gateway to future study of mathematics for many students. Despite the increasing societal need for individuals trained in mathematics, science, and engineering, calculus impedes many students' progress in these fields. The problems with calculus are clearly stated in the following excerpt:

Beginning with a conference at Tulane University in January 1986, there developed in the mathematics community a sense that calculus was not being taught in a way befitting a subject that was at once the culmination of the secondary mathematics curriculum and the gateway to collegiate science and mathematics. Far too many students who started the course were failing to complete it with a grade of C or better, and perhaps worse, an embarrassing number who did complete it professed either not to understand it or not to like it or both. For most students it was not a satisfying
culmination of their secondary preparation, and it was not a gateway for future work. It was an exit. (Dudley, 1993, p.vii)

Linn and Kessel (1995) examine students' attitudes toward calculus and higher mathematics courses. The students in the study consistently complain that calculus courses are designed to discourage future mathematics study. Large numbers of students switched out of mathematics after calculus; this was especially a concern for female students. Students blamed poor instruction for this decision. What is most striking about this study is that more often than not, it was the best students who chose to switch out of mathematics.

Annie Selden and her colleagues document many difficulties with problem solving in calculus. In one study, Selden, Mason, and Selden (1989) found that C students were unable to successfully complete nonroutine calculus problems. In a second study, Selden, Selden, and Mason (1994) discovered that A students were unable to complete nonroutine calculus problems. In a third study, Selden et al. (1999) determined that differential equations students who had successfully completed the year and a half calculus sequence were unable to solve nonroutine calculus problems. In each study, the authors recognize that students possessed the required knowledge in calculus but were unable to apply it to the nonroutine problems.

Another concern is that so few students successfully complete calculus. The American Mathematical Association for Two-Year Colleges (1995) published data that said only about 40% of students who intend to complete calculus are successful. Many studies show a steady decline of students pursuing undergraduate degrees in
mathematics, science and engineering (Committee on the Undergraduate Program in Mathematics, 2004; National Research Council, 1989). If these trends are to change, the mathematics community must re-examine the calculus curriculum.

**Calculus Reform**

The Mathematics Association of America recognized this problem when they published *Calculus for a New Century* (Steen, 1987). The authors of this document claim that calculus courses must encourage active student participation, not passive participation in a lecture. While the higher education community examined the role of the teacher in the calculus curriculum, the K-12 community examined the role of the teacher in the school curriculum. The National Council of Teachers of Mathematics (1991) writes,

> The teacher of mathematics must consistently expect and encourage students to work independently or collaboratively to make sense of mathematics; take intellectual risks by raising questions and formulating conjectures; display a sense of mathematical competence by validating and supporting ideas with mathematical argument. (p. 18)

The National Research Council (1989) promotes a similar viewpoint:

> Teachers’ roles should include those of consultant, moderator, and interlocutor, not just presenter and authority. Classroom activities must encourage students to express their approaches both orally and in writing. Students must engage mathematics as a human activity; they must learn to work cooperatively in small teams to solve problems as well as to argue convincingly for their approach amid conflicting ideas and strategies. (p. 61)

The American Mathematical Association of Two-Year Colleges (1995) clarifies this emerging philosophy in five standards of pedagogy:
1. Mathematics faculty will model the use of appropriate technology in the teaching of mathematics so that students can benefit from the opportunities it presents as a medium of instruction.

2. Mathematics faculty will foster interactive learning through student writing, reading, speaking, and collaborative activities so that students can learn to work effectively in groups and communicate about mathematics both orally and in writing.

3. Mathematics faculty will actively involve students in meaningful mathematics problems that build upon their experiences, focus on broad mathematical themes, and build connections within branches of mathematics and between mathematics and other disciplines so that students will view mathematics as a connected whole relevant to their lives.

4. Mathematics faculty will model the use of multiple approaches – numerical, graphical, symbolic and verbal – to help students learn a variety of techniques for solving problems.

5. Mathematics faculty will provide learning activities, including projects, and apprenticeships, that promote independent thinking and require sustained effort and time so that students will have the confidence to access and use needed mathematics and other technical information independently, to form conjectures from an array of specific examples, and to draw conclusions from general principles. (p. 15)

In tertiary mathematics, the result of these recommendations is the calculus reform curricula. Park and Travers (1996) write,

The themes of the calculus reform movement include: involving students in doing mathematics instead of lecturing at them; stressing conceptual understanding, rather than only computation; developing meaningful problem-solving abilities, not just 'plug-and-chug'; exploring patterns and relationships, instead of just memorizing formulas; becoming engaged in open-ended, discovery-type problems, rather than doing routine closed-form exercises; and approaching mathematics as a live exploratory subject, not merely a description of past work. (p.156)

In recent years, several calculus reform projects have been implemented around the country. The Harvard Project is one of the most influential calculus reform programs. The following excerpt from Hughes-Hallett (1997) demonstrates the goals of this curriculum:
Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures. Therein lies the danger of teaching calculus; it is possible to teach the subject as nothing but the rules and procedures – thereby losing sight of both the mathematics and of its practical value. . . . Our consortium set out to create a new calculus curriculum that would restore that insight. . . . Two principles guided our efforts . . . The Rule of Four: Every topic should be presented geometrically, numerically, algebraically and verbally. The Way of Archimedes: Formal definitions and procedures evolve from the investigation of practical problems”.

The Project CALC curriculum from Duke University is another influential calculus reform program. The goals of this curriculum include using calculus to formulate and solve real-world problems, using technology as an essential tool in the process, communicating meaning in both written and oral forms, distinguishing between and applying continuous and discrete models, and selecting between formal and approximate methods of solution (Smith & Moore, 1991).

A third influential calculus reform project is the Calculus and Mathematica curriculum from the University of Illinois. This technology-intensive approach requires students to complete interactive lessons that guide the students’ discovery of the key concepts of calculus. There is virtually no lecture time in this model. The teacher is no longer “curator of the dogma and arbiter of truth” (Brown, Porta & Uhl, 1991, p.100). Instead, the teacher assists the students in creating individual understanding.

A fourth influential calculus reform project is the Calculus, Concepts, Computers and Cooperative Learning (C4L) Curriculum from Purdue University. The following excerpt from the textbook describes the unique aspect of this curriculum:
You will write small pieces of code, or 'programs' that get the computer to perform various mathematical operations. In getting the computer to work the mathematics, you will more or less automatically learn how the mathematics works! Anytime you construct something on a computer then, whether you know it or not, you are constructing something in your mind. (Dubinsky, Schwingendorf & Mathews, 1995, pp. xiii)

As calculus reform has been implemented around the country, researchers have examined the effectiveness of the various projects. In examining the Harvard Calculus, Tidmore (1994) found that reform students did better on 11 questions and traditional students did better on two questions. In a follow-up study he examined a test of ten common questions. Four of these questions could be classified as reform oriented and six could be classified as traditional oriented. The results show that traditional students did better on one question and reform students did better on nine. In examining Project CALC, Bookman and Friedman (1994) found that students in the Project CALC curriculum outperformed students in a traditional curriculum on a test of problem solving. The authors also found that Project CALC students had better attitudes toward the calculus than the traditional students. In examining Calculus and Mathematica, Park and Travers (1996) found that students showed an increase in student conceptual attainment without showing any decrease in computational achievement. Also Calculus and Mathematica students showed improved attitudes toward the calculus. In examining Calculus, Concepts, Computers and Cooperative Learning (C4L), Schwingendorf, McCabe, and Kuhn (2000) found that C4L students earned higher calculus grades and were more inspired to continue studying calculus. The authors also found that C4L students were as adequately prepared for future studies in mathematics as other students, and
these students may be more adequately prepared for other academic courses that require calculus. Despite the generally positive research about the effects of calculus reform, some mathematicians have stated serious concerns about these curricula.

**Critics of Calculus Reform**

Wu (1996) challenges the lack of rigor in the text of the Harvard Project. Referring to the Hughes-Hallett description of the fundamental theorem of calculus, Wu (1996) claims, "When a seductively phrased heuristic argument, in reality very far from a proof, is presented without further comments, it is perilously close to a deception" (p. 1533). Wu later criticizes the Calculus and Mathematica program: "When these students first encounter on the software that the derivative of \( \sin x \) is \( \cos x \) rather than proving this statement the authors exclaim, 'How sweet it is. Math happens'" (Brown, Porta & Uhl, 1991, p. 103). Wu (1996) writes, "In other words, students are asked to believe that, thanks to the computer, they have witnessed mathematics at work" (p. 1533). Wu recognizes certain strengths of calculus reform. He appreciates the emphasis on making conjectures and examining counter-examples, yet he does not appreciate the “downplaying of symbolic computations, precise definitions, neat formulas, and precise answers” (p. 1534). He is also very concerned about the goal of making calculus accessible to students with limited algebra skills. He believes that this will result in less rigorous high school algebra classes.
Frank Allen, former president of the National Council of Teachers of Mathematics (1962-1964) had some very strong comments about the role of reform in the curriculum. The following statements appeared in an open letter to the president of the NCTM:

Reform! Don't we wish that our students could again attain the higher performance levels they reached in the 40s, 50s and early 60s? Instead of talking about "reform" we should be talking about regaining lost ground.

In his letter, he quotes a colleague:

Indeed, methods and gimmicks are a popular cop-out in teachers education programs. Universities seem to produce teachers who cannot understand the theory, research or principles underlying their subject, but rather want methods and techniques to satisfy and pacify their charges. (Allen, 1995)

Despite the criticisms, calculus reform curricula continue to influence how calculus is taught and how it is learned. Recently, the mathematics education community has begun to assess the success of the calculus reform movement.

Tucker and Leitzel (1995) examined the calculus curricula at 62 institutions. The authors found that little had changed. Most schools continued to teach calculus as a collection of procedures designed to meet the needs of mathematics, physical science and engineering students. Interestingly, students in these majors constituted a minority of the students enrolled in the calculus. Most schools made an effort to include some elements of reform such as increased use of technology and cooperative learning, but broad adoption of the philosophy of calculus reform had not taken hold.
Tucker and Leitzel (1995) discuss several impediments to calculus reform curricula. First, traditional homework assignments tell students that all problems should be completed quickly and difficulties can be remedied by reviewing the previous examples. Second, in order to implement a calculus reform curriculum an instructor must share and support the goals of the curriculum. Changing attitudes and beliefs would require considerable time and research. Without these changes in beliefs, changes in the curriculum are doomed to be superficial and short-lived. Third, most of the studies on calculus reform suggest general methods of instruction such as technology, collaboration, communication, open-ended problems and multiple representations. These studies did not sufficiently address how these techniques could be used with the key topics in calculus such as limits, derivatives and integrals.

There is great debate about how mathematics should be taught but there seems to be universal agreement that it must be taught in a more effective manner. Unfortunately there is no consensus on how that is to be done. In calculus there are many complicated topics. In order to improve calculus teaching and learning it seems reasonable to begin by examining the first substantive topic studied in calculus, the limit.

Concept of Limit

The limit is typically the first topic studied in the calculus curriculum. Cornu (1991) discusses the importance of the limit in the following excerpt:
The mathematical concept of a limit is a particularly difficult notion, typical of the kind of thought required in advanced mathematics. It holds a central position, which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity, and of differential and integral calculus. (p. 153)

Despite its importance, students struggle with the concept of limit. In fact teachers with several years of experience teaching calculus also struggle with the concept of limit (Simonsen, 1995). In order to understand why so many struggle with this concept, Cornu (1991) examined the historical development of this concept. Mathematicians encountered many difficulties in its development. Cornu lists (a) failure to connect geometric reasoning to numerical reasoning, (b) the difficult concepts of infinitely large and infinitely small, (c) the metaphysical difficulties with limits, and (d) the conceptually difficult notion of whether or not a limit is obtained. In response to these difficulties, Norman and Prichard (1994) write,

We wonder why, if it took mathematicians such a long time to formalize the notion of limit, we should expect students to understand adequately the rather unmotivated formalized version presented in calculus courses – and in one class period at that. (p. 74)

One of the reasons students struggle is that their personal image of the concept of limit differs from the formal definition. The relationship between the concept image of limit and the concept definition of limit helps describe this difficulty.
**Concept Image and Concept Definition**

Tall and Vinner (1981) define a concept image as the total cognitive structure that is associated with the concept, which includes all of the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (p. 152)

The concept definition is "a form of words used to specify that concept" (p.152). However the definition of a concept may be wholly incompatible with the concept image. This becomes problematic if the student fails to recognize the incompatibility (Tall and Vinner, 1981).

Based on student difficulties, Tall (1992) argues that it is inappropriate to approach the concept of limit from the perspective of the formal definition. Tucker (1986) does not include the formal definition of limit in a recommended syllabus for use in beginning college calculus. In order to design a curriculum that does meet the needs of the students, one must examine the concept images that students possess.

Similar to the notion of “concept image,” Cornu (1991) recognizes that students possess intuitions, images, and colloquial meanings prior to learning a formal concept like the limit. These informal ideas do not disappear after studying formal mathematics; rather they continue to influence how a student understands a concept. Cornu (1983) discusses several meanings students give to the ideas “tends toward” and “limit.” Students see “tends toward” as meaning approaching a number with either (a) eventually staying away from the number, (b) never reaching the number, (c) just reaching the number. Students see “limit” as an
impassible limit that can be reached or an impassible limit that cannot be reached. Others see a limit as either a point one approaches without reaching it or a point one approaches and reaches. Some other perspectives of limit include (a) a higher or lower limit, (b) a maximum or minimum, (c) a constraint, as in “speed limit,” (d) the finish. Cornu (1983) refers to ideas as spontaneous conceptions that arise from ordinary experience. He also argues that these conceptions do not fade once students study the formal definitions. Frid (1994) describes students who define limit as a personal limitation or barrier. This reinforces the notion that a limit is not reached. Ferrini-Mundy and Graham (1994) describe a student who claims that 0.9999999... is not 1 because one can get very close but one never actually gets to 1. The research indicates that many believe that a limit is unreachable. Unfortunately, these incorrect “intuitive” ideas do not generally disappear with formal instruction (Williams, 1991).

Thompson (1994) discusses many of the difficulties that students have with learning calculus. He states that most students create an image of a function that is simply a short expression on the left side, an equal sign in the middle, and a long expression on the right side. This notion makes it difficult for students to make necessary conceptual connections. Thompson also writes that students perceive various elements of calculus as static. This belief makes it difficult to understand how change in one notion influences change in a second notion. He later describes difficulties that students have with rate of change. These students have difficulty conceptualizing a changing rate and instead use an average rate of change. Still
others did not possess a schema for average rate of change. Finally, Thompson discusses the difficulties that students have with notation. He writes,

When students did interpret notation, it often came as an afterthought, and they often tended to read into the notation what they wanted it to say, without questioning how what they actually wrote might be interpreted by another person. More often, though, students would not interpret the notation with which they worked, but would instead associate patterns of action with various notational configurations and then respond according to those internalized patterns of action. (p. 51)

Student Difficulties with the Concept of Limit


1. Students often have difficulties with the terms “limit,” “tends to,” “approaches,” and “as small as we please.” The common usage of these terms may confuse the formal mathematical meaning.

2. Limit problems are not solved by simply applying arithmetic or basic calculus. Students must make use of infinite processes that Tall claims are “surrounded by mystery.”

3. Students have exceptional difficulty with the process of a variable getting arbitrarily small. They often see this as an arbitrarily small variable quantity suggesting an infinitesimal concept. Formal calculus rarely discusses properties of infinitesimals despite this common student construction.
4. The notion of numbers getting arbitrarily large suggests infinite numbers, which violates the notion that calculus is the study of real numbers. Students wonder whether or not a limit can actually be reached. Students are confused by the idea, "What happens at infinity?"

In light of all the difficulties, one must ask how a teacher can help a student understand these concepts.

Students have many difficulties with the concept of limit. One of the most common difficulties is that students have a "static" rather than a "dynamic" view of the limit (Cornu, 1991; Sierpinska, 1987; Williams, 1991). Cottrill et al. (1996) have attempted to devise a strategy that will enable students to overcome some of these difficulties. Their attempt is called the genetic decomposition of a limit.

Genetic decomposition is defined as a possible set of mental constructions that can be used to develop understanding of a given mathematical concept.

The following appears in Cottrill et al. (1996) as the genetic decomposition of a limit:

1. The action of evaluating \( f \) at a single point \( x \) that is considered to be close to or even equal to \( a \).
2. The action of evaluating the function \( f \) at a few points, each successive point closer to \( a \) than was the previous point.
3. Construction of a coordinated schema as follows:
   a. Interiorization of the action of step 2 to construct a domain process in which \( x \) approaches \( a \).
   b. Construction of a range process in which \( y \) approaches \( L \).
   c. Coordination of a) and b) via \( f \). That is the function \( f \) is applied to the process of \( x \) approaching \( a \) to obtain the process of \( f(x) \) approaching \( L \).
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of 3) is encapsulated to become an object.

5. Reconstruct the processes of 3c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit. (p. 174)

Cottrill et al. (1996) claim that the primary difficulty with the notion of limit comes from the fact that students must coordinate two different processes: (a) $x$ is approaching $a$, (b) $f(x)$ is approaching $L$. The fact that these two processes occur simultaneously causes a great difficulty. Another problem with the notion of limit is that students must understand how quantification is used in the definition. So in other words, students must coordinate the two processes into a new process; they must develop a schema for quantification and they must encapsulate this into the concept of limit. This is certainly difficult for most students. Their beliefs about the nature of mathematics may be a primary reason for many of these difficulties.

Williams (2001) concludes that students have a collection of naïve beliefs about limits. These include (a) Zeno’s paradox, that is, one never really reaches the limit; (b) one finds a limit by dividing an interval into an infinite number of subintervals; and (c) functions must be monotone. The most striking fact from Williams is that these ideas seem to get stronger after formal instruction. Students did not dismiss these ideas when presented with counter-examples; rather they viewed these as minor exceptions not worthy of much attention.

Sierpinska (1987) discusses epistemological obstacles that students encounter when studying the concept of limit. The four main obstacles she
identifies are (a) the nature of scientific knowledge, (b) the concept of infinity, (c) the concept of function, and (d) the notion of a real number. Like Williams (2001), she states that counter-examples together with proofs are not enough to change students’ notions. Many of these students see mathematics simply as a collection of opinions that need not be altered with one or two contradictory examples.

Szydlik (2000) discusses the relationships between students’ mathematical beliefs and their understanding of the concept of limit. Students were categorized as either having internal sources of conviction or external sources of convictions. Those with external sources had more difficulties with the concept of limit. They tended to believe that limits were unreachable, had inappropriate definitions, and were unable to justify their limit calculations. Students with internal sources of convictions performed much better.

Roh (2005) discusses several student misconceptions for determining the limit of a series. These include: (a) a series continues endlessly so it has no limit, (b) a limit can be found by plugging infinity in for $n$ and evaluating algebraically, (c) the series gets close to the number but never actually gets there, (d) the series needs to get close to a number or arrive at a number, thus resulting in two limits for a series, (e) a sequence has a limit if differences between consecutive terms get smaller. She determines that the reversibility of the $e - N$ process is crucial in moving from intuitive misconceptions of the limit to a complete understanding of the formal definition of the limit.

Frid (1994) examines how students approach calculus and she puts them in three categories that reflect their beliefs: (a) collectors, (b) technicians, and (c)
connectors. Collectors and technicians were not interested in understanding mathematics and therefore were not bothered by examples that contradicted their informal notions. The connectors, like the students with internal sources of convictions in Szydlik (2000), were able to begin to understand the concept of limit.

Oehrtman (2002) defines five metaphors students use to help them understand the concepts of limit. The collapse metaphor requires a student to imagine one dimension of a geometric object decreasing to zero so that a lower dimensional object is perceived as a limit. For example, a line defined by two points becomes a single point when secants are used to approximate a tangent. The approximation metaphor enables a student to disregard errors or differences if they are extremely small. For example, the limit of a function exists as long as you can get "pretty close" to the number. The closeness metaphor is similar to the approximation metaphor except it requires that the student see the numbers as points on a line. Therefore, the limit exists if the space between the points is negligible. The infinity as number metaphor suggests that students included the concept of infinity as a number and applied rules of calculus and algebra. For example, when solving an improper integral, these students simply plugged the "number" infinity into the antiderivative, thus applying the fundamental theorem of calculus. The physical limitation metaphor suggests that students imagine a smallest physical size beyond which nothing can exist. The limit of a sequence exists because at some points the numbers are so small that its size cannot exist in the physical world.

Oehrtman (2002) concludes by stating that these metaphors are almost always
incorrect, yet they may be beneficial in helping students begin to understand the concepts.

Wahlberg (1998) uses writing assignments to assist students with conceptual understanding of the limit. The students who completed the writing assignments performed significantly better than those in a control group. Also, the experimental students began to demonstrate an object level of understanding on the concept of limit. In her analysis of the student writing, Wahlberg noticed a duality. For example, a student knows that an improper integral exists but she does not really believe that the area is finite since the function goes to infinity. Another student knows that \(0.99999\ldots\) must be equal to 1 in math class, yet he believes that it really must be smaller than one.

Parks (1995) compares students who use Mathematica as an aid in learning the limit concept to those who do not. Those in the Mathematica section outperformed the control group with respect to the formal definition of the limit. Parks concludes that Mathematica is beneficial because it encourages a wide variety of problem-solving strategies. It promotes deconstruction of the limit concept and it encourages active student participation.

Simonsen (1995) examined high school advanced-placement calculus teachers' perceptions of the concept of limit, the role of limit in the calculus curriculum, and how one should teach the limit concept. These teachers thought calculus is a linear collection of topics in which the limit is the fundamental component. They believed that an intuitive understanding of limit is essential for later study in calculus. However, they spent little time on developing this intuitive
notion in class. Rather, they spent significant time examining the epsilon-delta definition of the limit, as required by the Advanced Placement Exam.

Conclusion

Calculus is a gateway into many technological and scientific fields, yet it is an impediment for many students. The topic of limit is the first sophisticated concept one studies in calculus. Reflective abstraction together with its components of interiorization, coordination, generalization, encapsulation and reversal is a promising area of research. For these reasons, a study that initiates reflective abstraction in order to improve student understanding of the concept of limit is valuable.
CHAPTER 3

METHOD

Research Question

Can a curriculum that promotes reflective abstraction through individual, peer, instructor and curriculum initiates improve student performance on the concept of limit?

Working Definitions

In order to answer this question all of the relevant terms must be defined.

Following the work of Piaget, the Research in Undergraduate Mathematics Education Community (RUMEC) defines reflective abstraction as follows:

Reflective abstraction is a concept introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. Reflective abstraction by an individual proceeds from two mechanisms which are necessarily associated. They are projection unto a higher level of that which was derived from a lower level, and, secondly, reflection which reconstructs and reorganizes within a larger system what is transferred by projection. (DeVries, 2001)

Dubinsky (1991) defines the constructs of reflective abstraction as interiorization, coordination, encapsulation, generalization and reversal. For the sake of this study, the working definitions of these terms follow.
**Constructs of Reflective Abstraction**

**Interiorization**
A student performs the steps in a procedure. The student reflects on the procedure and begins to define a concept.

**Coordination**
A student examines two different processes and integrates them into a coordinated process that is used to analyze a mathematical concept.

**Encapsulation**
A student encapsulates a concept by constructing individual meaning.
Encapsulation is the act of personifying a concept. An abstract notion or a collection of abstract notions becomes meaningful to an individual.

**Generalization**
After an individual has encapsulated a notion, it is extended and applied to a wider collection of mathematical problems.

**Reversal**
A student constructs a new mathematical notion by reversing the steps of the original notion.
The goal of this study is to encourage students to engage in these categories of reflective abstraction about the notion of limit. Piaget (1972) claims that reflective abstraction is an individual activity; however, Cobb et al. (1997) describe the role of reflective discourse in the classroom community: they argue that one goal of classroom discourse is to initiate individual student reflection. Extending the theory of Cobb et al. (1997), this study clarifies classroom community initiates to include individual, peer, instructor, and curriculum initiates. These terms are defined as follows:

**Individual initiate**
A student spontaneously engages in reflective abstraction.

**Peer initiate**
A classmate challenges or questions an individual. This encourages the student to engage in reflective abstraction.

**Instructor initiate**
The instructor challenges or questions an individual. This encourages the student to engage in reflective abstraction.

**Curricular initiate**
Activities in the curriculum are designed to challenge and question students. These encourage the student to engage in reflective abstraction.
This study categorizes inferences of reflective abstraction based on the category (interiorization, coordination, encapsulation, generalization, reversal) versus the initiate (individual, peer, instructor, curriculum). Examples that occupy each cell of the matrix shown in Figure 1 were collected from the data.

<table>
<thead>
<tr>
<th></th>
<th>Interiorization</th>
<th>Coordination</th>
<th>Generalization</th>
<th>Encapsulation</th>
<th>Reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Individual</td>
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<td>Peer</td>
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<td>Instructor</td>
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<tr>
<td>Curriculum</td>
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<td></td>
</tr>
</tbody>
</table>

Figure 1. Reflective abstraction category versus initiate matrix.

Constructs Versus Initiates

In order to clarify meaning, the following examples describe the kind of evidence that could be placed into each of the twenty cells.

Individual Initiate and Interiorization

Student uses a talk-out strategy or journal writing. This demonstrates that the student has internalized a procedure. Interiorization is inferred in the following hypothetical excerpt:

I kept choosing numbers closer and closer to 2 and my answers became closer and closer to 5. When I chose numbers closer and closer to 3, I obtained
numbers closer and closer to 7. I soon recognized that if I chose numbers closer and closer to a number \( a \), I would get numbers closer and closer to \( 2a + 1 \). This shouldn’t surprise me because the function is \( f(x) = 2x + 1 \).

**Individual Initiate and Coordination**

Student uses a talk-out strategy or journal writing. This demonstrates that the student identifies the relationships among concepts. Coordination is inferred in the following hypothetical excerpt:

When I chose numbers closer and closer to 2 from below, my answers became closer and closer to 5. When I chose numbers closer and closer to 2 from above, my answers became closer and closer to 5 as well. When I choose numbers closer and closer to 3 from below, I get 7. When I choose numbers closer and closer to 3 from above, I also get 7. I hypothesize that I do not need to do both from below and from above. The numbers will always be the same.

**Individual Initiate and Encapsulation**

Student uses a talk-out strategy or journal writing. This demonstrates personal understanding of the concept. Encapsulation is inferred in the following hypothetical excerpt:

I have just discovered a counter-example. My example is mailing a letter and I choose numbers closer and closer to 2 ounces. From below it costs me 41 cents. If I choose numbers closer and closer to 2 ounces from above, it costs me 58 cents. It seems to matter whether I choose from below or from above. I have also
looked at the example \( f(x) = \frac{2}{x^2} \). If I approach zero from below, the numbers get increasingly large. If I approach zero from above, the numbers get increasingly large. I will make a hypothesis. If I approach a number \( a \) from below and the values approach a number \( k \), and if I approach a number \( a \) from above and the values equal \( k \), I can say that the limit as I approach \( a \) equals \( k \).

**Individual Initiate and Generalization**

Student uses a talk-out strategy or journal writing. This demonstrates that a student has extended a concept into a new domain. Generalization is inferred in the following hypothetical excerpt:

I have looked at lots of problems with polynomials. It seems that the limit as you approach a number \( a \) of the polynomial \( f(x) \) will always be \( f(a) \). This strategy did not work with the mail example because the mail function is not a polynomial and its graph is not connected.

**Individual Initiate and Reversal**

Student uses a talk-out strategy or journal writing. This demonstrates that a student has constructed a new concept by reversing the meaning of the original concept. Reversal is inferred in the following hypothetical excerpt:

I wondered if I could construct a polynomial so that as I approach 2 the answers approach 7. I recognized that \( f(2) = 7 \). So I decided to use a quadratic
polynomial (parabola) so \( f(x) = ax^2 + bx + c \), so \( 7 = 4a + 2b + c \). If I let 
\[ a = 2, \quad b = 3, \quad c = -3, \] 
I get the right answer. So \( f(x) = 2x^2 + 3x - 3 \).

**Peer Initiate and Interiorization**

Students are working together on a group project. Questions from one student help the group execute and understand a procedure. Interiorization is inferred.

Johnny: We chose the numbers 6.9, 6.99, 6.999 and 6.9999 and we kept getting numbers closer and closer to 15. How could we describe what is happening?

Jamie: We could say that as we approach 7 from below, the answers approach 15.

Jill: Or we could say the limit from below 7 of the function is 15.

**Peer Initiate and Coordination**

The following example shows how student questions could help the group coordinate the notion of a connected graph (continuity) with the notion of a limit. Coordination is inferred.

Johnny: How is the notion of continuity related to the notion of the limit?

Jill: Remember the mail problem graph was not connected.

Jamie: A function must have a limit at a point if the graph is going to be connected.
Jill: I do not think that is enough for the graph to be connected. We can construct a function that has a limit but is not connected. Think about the “open circle” graphs.

Johnny: So we must have a limit and no open circles?

Jamie: In that case the graph would be connected or continuous.

Peer Initiate and Encapsulation

The following example shows how the students encapsulate the notion of continuity at a point. The questions among the groupmates help the students develop a personal understanding of the concept of continuity.

Johnny: So based on all of our examples, what has to happen for a graph to be connected at $x = 2$?

Jill: You mean for a function to be continuous at 2?

Jamie: Well, I think the limit as you approach 2 from below must equal the limit as you approach 2 from above.

Jill: That means the limit has to exist.

Jamie: And the value of the function $f(2)$ must exist.

Johnny: Is that enough?

Jill: No. The limit and the function value must also be the same.

Johnny: Why not?

Jill: If the limit and the value of the function are different, then there would be a hole.

Johnny: If they are the same, the hole is filled in and it is continuous.
Jill: So can we state a rule for continuity?

Johnny: Sure. A function being continuous at a point means that the graph is connected. We can check this without a graph by looking at the function. If the limit exists and it equals the value of the function, then the function is continuous at that point.

Peer Initiate and Generalization

The following excerpt shows how questions help the students generalize from continuity at a point to continuous everywhere:

Johnny: Well, the mail function was not connected at 2 ounces, but it was connected at 1 ounce. So is this graph continuous?

Jamie: It is continuous at some points, but not all points.

Jill: Many functions are continuous at all points. Think of lines and parabolas.

Jamie: Those functions always have limits and no open circles.

Johnny: So how could we define “continuous everywhere” functions?

Jill: The continuous at a point definition is true for all the points in the function.

Peer Initiate and Reversal

The following excerpt shows how students construct a function that is not continuous. Reversal is inferred.

Johnny: Can we create a graph of a function that is not continuous?
Jill: Sure. Think of the mail equation. That was not connected at 2 ounces.

Jamie: How about a math equation, an \( f(x) \) equation?

Jill: We need an open circle. How can we get one of those?

Jamie: I think we need something to cancel out.

Jill: How about \( f(x) = \frac{x-2}{x-2} \)?

Johnny: That is not continuous at \( x = 2 \) because \( f(2) \) does not exist.

Jill: Yes, but the limit exists in this case.

Johnny: That is not enough.

**Instructor Initiate and Interiorization**

The teacher helps the student interiorize the procedure of approaching a limit from the left.


Teacher: So what do you think would happen if you chose numbers still closer to 7?

Student: I believe I would get numbers closer to 15.

Teacher: How could you summarize the exercise?

Student: As I choose numbers closer and closer to 7 from below, my answers get closer and closer to 15.
**Instructor Initiate and Coordination**

The teacher helps the student coordinate the notions of limit and infinity:

Student: When I chose 1.9, the answer was 199. When I chose 1.99, the answer was 1999. The numbers seem to get bigger and bigger.

Teacher: Will the numbers ever stop getting bigger?

Student: I think the numbers will grow infinitely large.

Teacher: Is infinity a number?

Student: I don't think so.

Teacher: Do the answers approach a real number as you approach 2 from below?

Student: No.

Teacher: What is the limit as you approach 2 from below?

Student: It doesn't have one.

Teacher: So how are the notions of limit and infinity related?

Student: If a function grows toward infinity, then the limit does not exist there.

**Instructor Initiate and Encapsulation**

The teacher helps the student encapsulate the notion of limit:

Teacher: What must be true for a limit to exist at \( x = 2 \)?

Student: The limits must exist.

Teacher: What limits?
Student: From the left and from the right.

Teacher: What must be true about them?

Student: They must be the same.

Teacher: What if a limit gets infinitely large?

Student: That is not a real number.

Teacher: So how would you formally define that the limit for a function \( f(x) \) exists at \( x = 2 \)?

Student: A limit will exist if the function approaches the same real number, not infinity, as you approach 2 from the left and from the right.

**Instructor Initiate and Generalization**

The teacher helps the student to generalize his limit strategy to a class of functions:

Student: I am tired of plugging in numbers closer and closer to 2. Why can’t I just plug the number 2 into the function?

Teacher: Do you think that will always work?

Student: Yes, well... not the mail question.

Teacher: Is there a certain class of functions where it will work?

Student: \( f(x) \) questions.

Teacher: Well, what about \( f(x) = \frac{3+x}{x-2} \)?

Student: Well, it wouldn’t work at 2. You would have to divide by zero.

What about functions which never divide by zero?
Teacher: Can you think of what the graphs would like?

Student: Lines, parabolas, sine, cosine.

Teacher: What do those functions have in common?

Student: They are connected, continuous everywhere.

Teacher: How could you find the limits for continuous functions?

Student: I think I could just plug the numbers into the function.

Teacher: Can you explain why that must be true?

**Instructor Initiate and Reversal**

The teacher asks the student to create a counter-example demonstrating that it is not always fair simply to plug numbers into the function to evaluate the limit.

Student: I am tired of plugging in numbers closer and closer to 2. Why can’t I just plug the number 2 into the function?

Teacher: Can you create a function where it is not fair to plug 2 into it?

Student: Sure. $f(x) = \frac{1}{x - 2}$.

Teacher: So what is the limit as you approach 2 from the left?

Student: It gets large. The numbers go to negative infinity.

Teacher: Would the limit exist in that case?

Student: No. So the plug-in rule doesn’t always work.
Curriculum Initiate and Interiorization

A homework question that asks a student to perform a procedure and reflect upon that procedure would initiate interiorization. The following example demonstrates this.

Given the function \( f(x) = \frac{2}{(x-3)^2} \), plug in numbers closer and closer to 3 from below. Describe the process and its results in a succinct manner.

Curriculum Initiate and Coordination

A homework question that asks students to coordinate several ideas to solve a problem would initiate coordination. The following examples demonstrate this.

Does the limit from the right always equal the limit from the left? If possible construct an example where it does not occur.

If the limit from the left equals the limit from the right, must the graph be connected at that point? If possible, construct a graph of a nonconnected graph where the limits from above and below are the same.

If the limit from the left does not equal the limit from the right, could the graph be connected at the point? Try to construct such an example.

Curriculum Initiate and Encapsulation

A homework question that asks a student to describe personal understanding of a concept would initiate encapsulation. The following example demonstrates this.
A function is continuous at \( x = 2 \). We know that the graph is connected at \( x = 2 \). Construct a personal definition of continuity using the previous information.

**Curriculum Initiate and Generalization**

A homework question that asks a student to extend a notion into another context can initiate generalization. The following example demonstrates this.

Which trigonometric functions are always continuous? Which trigonometric functions are not always continuous? How does your personal definition of continuity relate to these groups of functions?

**Curriculum Initiate and Reversal**

A homework question that asks a student to reverse a concept to construct a new concept would initiate reversal. The following example demonstrates this.

Can you construct a function that is never continuous? How does this function relate to your formal definition of continuity?

**Descriptions of Curricula**

The experimental curriculum is modeled using the ACE (Activity, Classroom discussion, Exercise) teaching cycle (Weller et al., 2003). The activity is designed to initiate reflective abstraction through curricular initiates. Students work on the activity in cooperative groups that may promote reflective abstraction through peer initiates. The instructor de-briefs the activity in a classroom discussion that may promote reflective abstraction through instructor initiates. Each individual
completes a set of exercises related to the activity providing an opportunity for reflective abstraction through individual initiates.

The traditional curriculum uses *Calculus* by Larson, Hostetler, and Edwards (2006) as a guide. The instructor demonstrates examples similar to those in the text as part of the lecture. Students complete standard text exercises in class. Students also complete standard text exercises for homework. The instructor begins each subsequent day by answering student questions about the homework.

**Experimental Design**

A great difficulty in performing an experiment of this type is accounting for innate differences in the experimental and control groups. In an experimental design, individuals would be assigned to these groups randomly or systematically using information such as ACT scores in order to minimize these differences. This would have been difficult to do for this study. Rarely are two sections of Calculus I offered at the same time on the same days. Therefore, assignment to control and treatment groups occurred based on student self-enrollment patterns. In order to minimize a time factor, two sections were examined that met at roughly the same time of day and for the same number of days per week.

In order to “equalize” the groups, the researcher planned to eliminate individuals that were extremely different from typical community college calculus students. The following questionnaire was used to identify student backgrounds.
1. Have you enrolled in calculus before? If so, describe the course, institution, grades, etc.

2. Have you completed a college course in trigonometry? If so, describe the course, institution, grades, etc.

3. Have you completed a college course in college algebra/pre-calculus? If so, describe the course, institution, grades, etc.

4. Have you completed a developmental mathematics course while in college? If so, describe the course, institution, grades, etc.

5. How many years of high school mathematics did you complete? What high school mathematics courses did you complete? Did you complete high school more than 5 years ago?

6. Did you take the college placement test in mathematics? If so, what was your score?

7. Did you take the ACT or SAT exam? If so, what was your mathematics sub-score?

Threats to Validity

Dawson (1997) discusses eight threats to internal validity:

1. History: Environmental events occurring between first and second observations in addition to the independent variable.

2. Maturation: Change due to the passage of time, not the independent variable.

3. Testing: Sensitization to the posttest as a result of taking the pretest.
4. Instrumentation: Deterioration or changes in the accuracy of the instruments or observations used to measure the dependent variable.

5. Statistical regression: Extreme scores tend to regress toward the mean on repeated testing.

6. Selection: Placing participants in certain groups based on preferences.

7. Mortality: Loss of participants and their data due to various reasons.


Dawson (1997) also discusses five threats to external validity:

1. Interaction of selection and treatment: An effect between a treatment and a certain other group may not be generalized to hold for a different group.

2. Interaction of setting and treatment: Can a relationship on a military base also be obtained on a university campus?

3. Interaction of history and treatment. If the experiment were conducted a day after a traumatic event, the results should not be generalized to the following week (Campbell & Stanley, 1963; Cook & Campbell, 1979).

4. Interaction of treatments with treatments: Multiple treatments administered to the same subjects may result in cumulative effects.

5. Interaction of testing with treatment: The pretest may increase or decrease the subjects’ responsiveness to the posttest (Parker, 1993).

In order to minimize the innate differences in the two groups, a pretest-posttest model with an experimental group and a control group is used. This model answers most of Dawson’s threats.
1. History.

The pretest and posttest were given after a relatively short period of time. Any environmental changes in the experimental group were likely be mirrored in the control group.

2. Maturation.

Students at this level may mature with respect to study skills during the first weeks of college calculus. Again, changes in one group should be reflected in changes in the other.

3. Testing.

Both groups were given the pretest. I recognized that sensitization would be likely, but it should happen in both groups.

4. Instrumentation.

Similar questions were used on both the pretest and the posttest. Similar rubrics were used to grade them.

5. Statistical regression.

This is a problem; however, it is no more likely to happen in one group than the other.

Students enrolled in the respective sections using standard enrollment patterns. Teachers were chosen based on similar experience and desire to participate in the project.

7. Mortality.

This is a real problem in college calculus classes. I hoped that the mortality rates in the two sections would be similar. I tried to minimize this by the one-week length of the study.

8. Interaction.

This problem should influence both sections equally.

The following answer the threats to external validity:


Selection of students is by usual enrollment patterns.


The students represent community college calculus students in an affluent suburban district. It seems reasonable to generalize the results to similar groups. Future studies would be needed to determine if results could be generalized to students in high schools or highly selective universities.


The history for the two groups was presumed to be similar, so it should not have been an issue.

4. Interaction of treatments with treatments.

There were a small number of treatments in this study, so this interaction was not a concern.
5. Interaction of testing with treatment.

This was a realistic concern; however it would have affected the control group and the treatment group in a similar fashion.

Selection of Participants

Teachers

The researcher selected two community college mathematics teachers at the same institution to participate in the project. The first goal of the selection process was to select individuals with similar levels of experience. The second goal of the selection process was to select individuals who teach calculus courses with similar schedules. The third goal was to select individuals who regularly participate in activities designed to improve instruction.

At the time of the study, the first teacher had been teaching calculus for six years. She uses technology on an inconsistent basis but has shown an interest in increasing her use of technology. She regularly teaches the first- and second-semester calculus classes and she regularly teaches differential equations classes. In addition to teaching calculus-based classes, she often teaches classes at the developmental level. Her undergraduate and graduate degrees are in mathematics.

The second teacher had been teaching calculus for six years. He rarely uses technology but has expressed an interest in learning how to use it in calculus. He regularly teaches first-, second- and third-semester calculus classes as well as
differential equations classes. Like the first teacher, he teaches many developmental classes. His undergraduate and graduate degrees are in mathematics. He has earned a terminal degree in mathematics.

First-semester calculus courses are offered using many models at this institution: (a) five days per week and 50 minutes per day, (b) three days per week and 85 minutes per day, (c) two days per week and 125 minutes per day. Classes are also offered both during the day and during the evenings. These student groups are very different. Day students tend to be traditional-age students (18-23) with few older students. Evening students are older with few traditional-age students. In order to compare similar classes, the researcher chose teachers who teach day classes that meet five days per week, 50 minutes per day.

Both teachers regularly participate in activities designed to improve teaching. The first teacher has attended many Great Teacher retreats offered by the college. This three-day retreat enables faculty from diverse disciplines to discuss effective teaching strategies. Both teachers have participated in workshops offered by the Teaching and Learning Center at the college. The first teacher recently completed an online graduate class from Portland State University on the use of graphing calculators in algebra class. The second teacher consistently attends lectures on teaching and learning. Both teachers regularly work in the math assistance area, a drop-in service for students with questions.

Schedules, classroom experiences, and participation in learning activities indicate that these teachers are comparable. Most importantly, these teachers wanted to participate. The researcher recognizes that it is impossible to select two
identical teachers; however, these teachers share many similarities. It may be fair to claim that the teacher effect in this study was not exceptionally large.

**Students**

There are several sections of Calculus I offered each term at the college. Students enrolled in the experimental section and the control section participated in the quantitative analysis. Based on the results of the quantitative analysis, students were chosen to participate in the qualitative portion of the study.

**Description of Quantitative Analysis**

The goal of the quantitative analysis is to identify differences between the experimental group and the control group. The goal of the qualitative analysis is to identify the causes for these differences.

**Pretest – Posttest Analysis**

All pretests and posttests were scored using a 2-point rubric designed by the Illinois State Board of Education (2005b):

2: Completely correct response, including correct work shown.
1: Partially correct response.
0: No response, or the response was incorrect.

The pretests for the experimental and control groups were examined to see if there was a significant difference between the means of the two groups. The posttests were scored to determine whether or not one group of students
outperformed the other group of students. An independent grader used the defined rubric to score the pretests and posttests in order to establish inter-rater reliability. In the event of discrepancies in scoring, the researcher and the independent grader further defined and refined item-specific scoring schemes.

Description of Qualitative Analysis

Twelve students were chosen to participate in the qualitative study. The results of the quantitative analysis determined the students for the qualitative study. Two students from the experimental section who showed the greatest improvement from pretest to posttest and two students from the control section who showed the greatest improvement were also selected. Two students from the experimental section who showed median improvement from pretest to posttest and two students from the control section who showed median improvement were selected. Finally two students from the experimental section who showed little improvement and two students from the control section who showed little improvement were selected. The twelve students were classified into subgroups based on their performance relative to the median on the pretest and their performance relative to the median on the posttest. The students who scored below the median on the pretest and above the median on the posttest were classified as the Improve subgroup. The researcher compared the performance of students in this group to students in groups with little or no improvement. This analysis helped clarify the characteristics of successful students.
The researcher re-examined the posttests used in the quantitative study for evidence of reflective abstraction. These tests were scored using the rubric developed by the Illinois State Board of Education (2005a). It was adapted from Lane (1993). This rubric is effective because it requires separate scores for mathematical knowledge, strategic knowledge and explanation. Each of the twelve students received a set of three scores from this rubric. These scores were analyzed to see if there was a difference of performance levels between the six experimental section students and six control section students on the measures of mathematical knowledge, strategic knowledge or communication.

**Interviews**

The researcher interviewed each of the students in the qualitative study. The goal of these interviews was to infer to what extent the students engaged in reflective abstraction while solving problems. The researcher conducted the interviews using a modified version of the standardized open-ended interview as defined by Patton (1990). Patton (1990) describes the standardized open-ended interview in the following excerpt:

> A set of questions carefully worded and arranged with the intention of taking each respondent through the same sequence and asking each respondent the same questions with essentially the same words. Flexibility in probing is more or less limited, depending on the nature of the interview. (p. 198)

Patton claims that the primary advantage of this type of interview is that all students answer the same questions, so it is appropriate to compare responses. In addition, this interview structure also helps in organizing and analyzing the data.
Wahlberg (1998) describes the limitations with this type of interview:

“There is little flexibility in relating to particular individuals, and the standardized wording of questions may constrain the naturalness and relevance of questions and answers” (p. 65). In order to correct this difficulty, Wahlberg asks all of the questions to all of the students, but she allows some flexibility in adding additional questions or changing the order of the questions in order to meet the unique needs of the students. One of her strategies is if a student does not answer a question after 30 seconds, she would rephrase the question or provide a gentle prompt.

Douglas (1985) discusses the necessary flexibility in the interview process in the following statement: “Creative interviewing . . . involves the use of many strategies and tactics of interaction, largely based on an understanding of friendly feelings and intimacy, to optimize cooperative mutual discourse and a creative search for mutual understanding” (p. 25). The interview is designed to identify the various constructs of reflective abstraction that the students use. In this study, the students were asked to clarify their procedures and conceptual understanding of the problems on the posttest. The researcher provided the student a copy of the original question and the student's solution. The researcher asked the following questions to each of the students.

Question 1: State in your own words what this question means to you.

Question 2: Describe how you solved this problem.

Question 3: What are the key concepts described in this question?

Question 4: What does this concept mean to you?

Question 5: How are these concepts related to other concepts studied in this unit?
The posttests and interviews were coded using the five constructs of reflective abstraction. The researcher inferred a certain number of interiorization, coordination, encapsulation, generalization and reversal examples for each of the twelve students. A tally of the number of inferences were examined to see if there was a difference among the performances of high-improvement, median-improvement and low-improvement students. These tallies were also analyzed to see if there was a difference between the students in the experimental section and the control section.

Data Sources

In addition to pretests, posttests, and interviews, additional data sources included homework sets, classwork sets, audiotaped groupwork sessions, and audiotaped class sessions. In particular, the researcher observed the experimental class and the control class in order to verify how the curricula and the teachers initiate reflective abstraction. Also, the researcher monitored and audiotaped group activities in order to infer how peers can initiate reflective abstraction. Finally, the researcher examined homework and classwork assignments to infer how the curriculum and the individual can initiate reflective abstraction.

The data sources including homework sets, class observations, audiotaped group sessions, posttests and interviews were analyzed in order to infer how the students engaged in reflective abstraction. The goal of this data was to demonstrate to what degree the experimental and control sections promoted reflective abstraction. The data sources were examined to document the initiates that
promoted the various constructs of reflective abstraction. A collection of
frequencies of inferences of reflective abstraction were coded by the construct and
initiate. Examples were identified in each of the cells as shown in Figure 1.

**Directions to the Teachers**

The researcher met with each teacher individually and explained to the
teachers that the experiment was designed to measure the effects of a curriculum on
student understanding of the concept of limit. Each teacher was given a copy of the
respective curriculum and told to keep deviations to a minimum. The researcher
designed all of the in-class activities and all of the homework activities for each
curriculum. In order to keep external influences to a minimum (outside groupwork,
math assistance area, tutors), students were told to complete all homework
assignments independently.

The teachers were asked to read the following statement before the
implementation of the experiment: "We are about to begin a study on how students
learn calculus. It is very important that you attend class each day so that we can
obtain reliable data. Please take this seriously. Give it your best effort. With your
assistance, the data from this experiment can help other students learn calculus more
effectively."

The teacher assigned to the experimental curriculum was be told that this
curriculum is designed to improve student performance and understanding on the
concept of limit. Students were to improve their conceptual understanding of limit
rather than simply memorize definitions and procedures. This teacher was
instructed to keep "teacher-telling" to a minimum and asked to tell the students to
ignore the textbook for this part of the course. The researcher told the teacher that it
is important that students reflect on their learning rather than getting validation from
the teacher or the textbook.

The teacher assigned to the control curriculum was told this curriculum is
designed to improve student performance and understanding on the concept of limit.
The model of this curriculum was (a) definition, (b) teacher example, (c) seat-work
exercise, (d) teacher completes exercise, (e) teacher summarizes the topic. The
researcher provided the lessons for this curriculum. This instructor was told to
encourage students to use the textbook for completing in-class and homework
activities.
CHAPTER 4
RESULTS

Introduction

The collection and analysis of the data were designed to answer the question, “Can a curriculum that promotes reflective abstraction through individual, peer, instructor and curriculum initiates improve student performance on the concept of limit?” In order to answer the question, two sections of Calculus I students participated in the study. The first group of students studied a traditional curriculum (Control) and the second group of students studied a curriculum designed to promote reflective abstraction (Experimental). The quantitative data includes student scores on pretests and posttests. This data provides evidence that the students in the experimental section outperformed the students in the control section. The qualitative data includes interviews, observations, and protocols. This data provides evidence of why the experimental students may have outperformed the control students.

Quantitative Data

The researcher and an independent grader scored the pretests and posttests using the short-response rubric from the Illinois State Board of Education (2005b).
There were very few discrepancies in grading. These discrepancies were remedied by constructing and consulting item-specific scoring schemes and re-grading unscored tests.

**Pretest Analysis**

The pretests consisted of twelve computational questions. Students earned 2 points for a correct solution, 1 point for a partially correct solution, and 0 points for an incorrect solution. A total of 35 students participated. Table 1 summarizes the statistics from the pretests. Using a two-tailed t test and a significance level of $p < .05$, the results were $t(32) = 0.57$ and $p = .58$. So no significant difference was found between the means of the control group and the experimental group on the pretests.

Table 1

**Scores on Pretest – All Participating Students**

<table>
<thead>
<tr>
<th>Measure</th>
<th>Experimental</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>16</td>
<td>19</td>
</tr>
<tr>
<td>Mean</td>
<td>7.313</td>
<td>6.316</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>5.237</td>
<td>5.132</td>
</tr>
</tbody>
</table>
Posttest Analysis

The posttests consisted of 14 questions. Twelve questions were computational and two questions were essays. These tests were scored using the same short-response rubric from the Illinois State Board of Education (2005b) that was used in the pretest analysis. Table 2 summarizes the posttest scores for all students who participated in the study. Using a significance level of \( p < .05 \), a one-tailed \( t \) test for equality of means was performed. The results of the \( t \) test are \( t(32) = 2.63 \) and \( p < .01 \). This demonstrates that the students in the experimental section outperformed those in the control section.

Table 2

Scores on Posttest – All Participating Students

<table>
<thead>
<tr>
<th>Section</th>
<th>Experimental</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>16</td>
<td>19</td>
</tr>
<tr>
<td>Mean</td>
<td>21.18</td>
<td>16.95</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>4.48</td>
<td>5.14</td>
</tr>
</tbody>
</table>
A small number of students missed at least one of the five class sessions. A second analysis of the posttests examined the performance of the students who attended all of the class sessions. Table 3 summarizes the posttest scores for the students who attended all classes. Using a significance level of $p < .05$, a one-tailed $t$ test for equality of means was performed. The results of the $t$ test are $t(25) = 1.76$ and $p = .046$. This demonstrates that the students in the experimental section outperformed those in the control section.

Table 3
Scores on Posttest Restricted to Students Who Attended All Classes

<table>
<thead>
<tr>
<th>Section</th>
<th>Experimental</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>Mean</td>
<td>21.38</td>
<td>18.4</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>4.50</td>
<td>4.47</td>
</tr>
</tbody>
</table>

In order to minimize differences that might be due to previous knowledge, an analysis of covariance is included. The posttest scores are covaried against the pretest scores. The results are $F = 6.40$, $p = .017$. Again using a significance level...
of $p < .05$, a significant difference is present. All of the results are summarized in Table 4.

Table 4

Posttests Versus Section Type Covaried with Pretest Score for All Students

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Seq SS</th>
<th>Adj SS</th>
<th>Adj MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>1</td>
<td>148.25</td>
<td>121.03</td>
<td>121.03</td>
<td>6.01</td>
<td>0.020</td>
</tr>
<tr>
<td>SectionType</td>
<td>1</td>
<td>128.94</td>
<td>128.94</td>
<td>128.94</td>
<td>6.40</td>
<td>0.017</td>
</tr>
<tr>
<td>Error</td>
<td>32</td>
<td>644.36</td>
<td>644.36</td>
<td>20.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>34</td>
<td>921.54</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Qualitative Data

In an attempt to better understand the thought processes of students, a subsequent qualitative analysis was implemented. It was predetermined that twelve students would participate in the qualitative analysis. These twelve students form the three-tiered comparison subgroup. Six students were selected from the experimental section and six students were selected from the control section. These students were classified into three categories: (a) Improve, (b) Maintain, and (c) Regress. The Improve category consists of six students who scored below the
median on the pretest and at or above the median on the posttest. The Maintain category consists of two students who scored above the median on both the pretest and the posttest. The Regress category consists of four students who scored above the median on the pretest but scored at the median level or below on the posttest. The qualitative analysis will suggest why some of these students improved and others did not.

**Interview Analysis**

Each of the 12 students in the three-tiered comparison subgroup was interviewed. Each student was asked to clarify answers to the questions on the posttests. The interviews were analyzed to infer when and how often students engaged in each construct of reflective abstraction. The five constructs of reflective abstraction are interiorization, coordination, encapsulation, generalization and reversal. The working definitions of the terms follow.

1. **Interiorization**: A student performs the steps in a procedure. The student reflects on the procedure and begins to define a concept.

2. **Coordination**: A student examines two different processes and integrates them into a coordinated process that is used to analyze a mathematical concept.

3. **Encapsulation**: A student encapsulates a concept by constructing individual meaning. Encapsulation is the act of personifying a concept. An abstract notion or a collection of abstract notions becomes meaningful to an individual.
4. Generalization: After an individual has encapsulated a notion, it is extended and applied to a wider collection of mathematical problems.

5. Reversal: A student constructs a new mathematical notion by reversing the steps of the original notion.

Each of the twelve interview transcripts was coded according to the examples of interiorization, coordination, encapsulation, generalization and reversal. A list of these examples appears in Table 5. The examples in Table 5 were used to help categorize the inferences of reflective abstraction demonstrated by each student in the interviews.

The interviews for the six students in the Improve group were analyzed and the inferences of the constructs of reflective abstraction were tallied. The number of inferences for each construct appears in Table 6. The totals indicate that these students demonstrated coordination and interiorization most often. The totals also reflect a fairly large number of generalization and reversal inferences. Encapsulation was rarely inferred. These tallies appear in Table 6.

The interviews for the two students in the Maintain group were analyzed and the inferences of the constructs of reflective abstraction were tallied. Like the students in the Improve group, coordination was inferred most frequently. Unlike the students in the Improve group, generalization and reversal were rarely inferred. These tallies appear in Table 7.
<table>
<thead>
<tr>
<th>Categories</th>
<th>Examples</th>
</tr>
</thead>
</table>
| **Interiorization** | Using the polynomial substitution rule to evaluate a limit.  
Using a procedure to find a limit graphically.  
Using an algebraic process of finding the limit.  
Interiorization of a process of evaluating a limit.  
Develop an algebraic process for finding an asymptote.  
Develop a process that demonstrates that the limit does not exist.  
Execute a strategy for establishing continuity of a piecewise function. |
| **Coordination** | Coordinating the notions of left-hand limits, right-hand limits, asymptotes and limits.  
Coordinating notions of left-hand limits, right-hand limits and limits.  
Coordinating use of tables and substitution.  
Coordinating a graphical perspective and an algebraic perspective.  
Coordinating notions of limit and continuity.  
Coordinating the notions of continuity, left-hand limits and right-hand limits.  
Coordinating notions of limit, continuity, and removable discontinuity.  
Coordinating notions of limit, continuity, division by zero and removable discontinuity.  
Coordinating the notions of continuity and division by zero.  
Coordinating algebraic simplification (all-but-one point rule) and types of discontinuities.  
Coordinating notions of algebraic simplification, asymptotes, and limit.  
Coordinating table values and asymptotic behavior.  
Coordinating algebraic simplification and non-removable discontinuity.  
Coordinating the algebraic process of cancellation and the removable discontinuity.  
Coordinating the notions of continuity and algebraic simplification.  
Coordinating values from a table and asymptotic behavior.  
Coordinating limits and asymptotic behavior.  
Coordinating a graphical representation, the left-hand limit, the right-hand limit and the limit.  
Coordinating table values and limits.  
Coordinating a graphical representation, the left-hand limit, the right-hand limit and the limit and removable discontinuity.  
Coordinating notions of limit, continuity, division by zero. |
| **Encapsulation** | Personal understanding of the concept of limit.  
Personal understanding of the notion of continuity. |
| **Generalization** | Generalizing from an algebraic representation to a geometric representation.  
Generalizing left-hand and right-hand limits to piecewise functions.  
Creating a rule.  
Generalizing the polynomial substitution rule to rational function.  
Identifying uses of the limit.  
Construct an example that has a limit but is not continuous.  
Generalizing from algebraic procedure to removable discontinuity.  
Generalizing left-hand and right-hand limits to piecewise functions.  
Extending the substitution rule to quotients.  
Extending limit notion to real-world context.  
Extending algebraic simplification to the indeterminate form. |
| **Reversal** | Reversing the definition of continuity.  
If a function is not continuous, there may not be a value of the function.  
Reversing the definition of limit to explain when a limit does not exist.  
Reversing the definition of limit to construct examples of where the limit does not exist.  
Reversing the substitution rule to show why it does not apply. |

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Table 6

Number of Examples of Constructs of Reflective Abstraction – Improve Group

<table>
<thead>
<tr>
<th>Name</th>
<th>Interiorization</th>
<th>Coordination</th>
<th>Generalization</th>
<th>Reversal</th>
<th>Encapsulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dan</td>
<td>5</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Sam</td>
<td>8</td>
<td>13</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Ron</td>
<td>2</td>
<td>15</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Bruce</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Chuck</td>
<td>9</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Karen</td>
<td>9</td>
<td>10</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Totals</td>
<td>40</td>
<td>63</td>
<td>20</td>
<td>23</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 7

**Number of Examples of Constructs of Reflective Abstraction – Maintain Group**

<table>
<thead>
<tr>
<th>Name</th>
<th>Interiorization</th>
<th>Coordination</th>
<th>Generalization</th>
<th>Reversal</th>
<th>Encapsulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Larry</td>
<td>6</td>
<td>16</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Maria</td>
<td>3</td>
<td>14</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Totals</td>
<td>9</td>
<td>30</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

The interviews for the four students in the Regress group were analyzed and the inferences of the constructs of reflective abstraction were tallied. Like the first two groups, coordination was inferred most often. Like the Maintain group, generalization was rarely inferred. However, reversal was more prevalent in the Regress group than it was in the Maintain group. These tallies appear in Table 10.
Table 8

Number of Examples of Constructs of Reflective Abstraction – Regress Group

<table>
<thead>
<tr>
<th>Name</th>
<th>Interiorization</th>
<th>Coordination</th>
<th>Generalization</th>
<th>Reversal</th>
<th>Encapsulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greg</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>John</td>
<td>4</td>
<td>8</td>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Vern</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Alice</td>
<td>3</td>
<td>15</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Totals</td>
<td>19</td>
<td>36</td>
<td>4</td>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

The six students in the Improve category demonstrated a total of 148 inferences of reflective abstraction for a mean of 24.67 per student. The two students in the Maintain category demonstrated a total of 46 inferences of reflective abstraction for a mean of 23.00 per student. The four students in the Regress category demonstrated a total of 73 inferences of reflective abstraction for a mean of 18.25 per student. One serious concern with the qualitative analysis is the rare inference of encapsulation.
Problems with Encapsulation

In describing encapsulation, Dubinsky and Lewin (1986) write, “It is only later (and it may not happen for everyone) that the epistemic subject sees the operation as a total structure. Reflective abstraction includes the act of reflecting on one’s cognitive action and coming to perceive the collection of thoughts as a structured whole. As a result, the subject can now encapsulate the structure, and see it as an aliment for other structures” (p. 63).

The curricula were conducted over a five-day period. In retrospect it may be unlikely that a complex cognitive structure like the limit could be fully encapsulated in this time frame. There is evidence in the protocols that certain students were beginning to construct a schema for limit and continuity. However, it is difficult to know whether or not a student has truly encapsulated the notion or is simply quoting memorized material.

There was one question in the experimental curriculum that was primarily focused on encapsulation. It asked, “What do limits mean to you?” The best response came from the following student:

A limit is when, as the function is getting closer and closer to the same x-value from the left and the right, the function is getting closer and closer to the same y-value from the left and from the right. Continuous graphs always have a limit for any x-value. You can draw a continuous graph without lifting your pencil from the paper. A graph that is not continuous, you have to lift your pencil from the paper to keep drawing it. Functions that have a limit even though they are not continuous reach the same number from the left and from the right even if there may or may not be a y-value. Examples include functions with removable discontinuities that create holes in graphs. You can use the all-but-one-point rule to find the limit. A function that is not continuous and has no limit is because the function gets closer to a
different number from the left and from the right. You can use tables or graphs to see this.

There are elements in her discussion that indicate that she is developing schemas for limits and continuity. She uses both formal and informal definitions and examples. She compares and contrasts the two notions. She extends the ideas to removable discontinuities. Her discussion includes aspects of interiorization, generalization, coordination and reversal. However, it may not yet be fair to conclude that her writing is fully indicative of encapsulation.

Using the working definition, there are very few examples of encapsulation in this study. The primary goal of both curricula may have been encapsulation of the limit concept, yet a five-day lesson may be too brief to see sufficient evidence.

**Extended Response Analysis**

In order to determine additional information from the twelve students in the three-tiered comparison subgroups, the posttests were re-examined. Six of these students were in the control section and six of these students were in the experimental section. A two-tailed, two-sample $t$ test examined the difference between the means of the two sections for the students in the comparison subgroup. The results are that $t(10) = 0.94$ and $p = .37$, so no significant difference at the $p < .05$ level was identified. The data is summarized in Table 9.
Table 9

Scores on Posttest – Comparison Subgroup

<table>
<thead>
<tr>
<th>Section</th>
<th>Experimental</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Mean</td>
<td>21.67</td>
<td>19.67</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>3.67</td>
<td>3.72</td>
</tr>
</tbody>
</table>

For further evidence, an analysis of covariance was performed using the pretest scores as a covariate. The results of that test show that $F = 2.54$ and $p = .17$ so the result was not significant at the $p < .05$ significance level. Therefore it may be fair to compare the six students in the experimental group to the six students in the control group. Additional results from the analysis of covariance are summarized in Table 10.
Table 10

Posttests Versus Section Type Covaried with Pretest Score for Comparison

Subgroup

Analysis of Variance for Posttest, using Adjusted SS for Tests

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Seq SS</th>
<th>Adj SS</th>
<th>Adj MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>1</td>
<td>25.27</td>
<td>16.57</td>
<td>16.57</td>
<td>1.51</td>
<td>0.250</td>
</tr>
<tr>
<td>Section Type</td>
<td>1</td>
<td>24.63</td>
<td>24.63</td>
<td>24.63</td>
<td>2.24</td>
<td>0.168</td>
</tr>
<tr>
<td>Error</td>
<td>9</td>
<td>98.76</td>
<td>98.76</td>
<td>10.97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>148.67</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The twelve posttests for the students in the three-tiered comparison subgroup were rescored using the five-point extended response rubric from the Illinois State Board of Education (2005a). The tests were scored three times: first for mathematical knowledge, second for strategic knowledge and third for explanation. The results appear in Table 11.
Table 11

Extended Response Results for Students in the Comparison Subgroup

<table>
<thead>
<tr>
<th>Measures</th>
<th>Mathematical Knowledge</th>
<th>Strategic Knowledge</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experimental Control</td>
<td>Experimental Control</td>
<td>Experimental Control</td>
</tr>
<tr>
<td>N</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>M</td>
<td>40.67</td>
<td>41.50</td>
<td>44.33</td>
</tr>
<tr>
<td>SD</td>
<td>5.20</td>
<td>6.28</td>
<td>7.09</td>
</tr>
</tbody>
</table>

One-sided $t$ tests for equality of means of the experimental and control students were performed for mathematical knowledge, strategic knowledge and explanation measures. A significance level of $p < .05$ was used for each of the tests. No significant difference was identified for the mathematical knowledge measure, $t(10) = 1.10, \ p = .15$. No significant difference was identified for the strategic knowledge measure, $t(10) = 1.31, \ p = .11$. A significant difference was identified for the communication measure, $t(10) = 1.98, \ p = .04$.

Initiates

The experimental curriculum was designed to initiate reflective abstraction. To demonstrate that this occurred, student class work, student homework and
transcripts of group work were examined. A sample of these protocols was collected from the three-tiered comparison subgroup. Items from these protocols were coded based on the construct of reflective abstraction (interiorization, coordination, generalization, reversal, and encapsulation) and the type of initiate (curriculum, peer, instructor, individual). Curriculum initiates were built into the experimental lessons. Peer initiates occurred as the students completed the assignments in groups. Instructor initiates occurred as the teacher asked questions that helped students clarify the concepts. Individual initiates occurred as the students completed the textbook homework.

The control curriculum was not designed to initiate reflective abstraction nor was it designed to hinder it. In order to determine how it occurs in a traditional classroom, a collection of lessons was audiotaped. The transcripts of these lessons were analyzed to identify instructor initiates of reflective abstraction. Peers were not given the opportunity in the classroom to work together, so there were no opportunities to infer peer initiates of reflective abstraction. Both the experimental section students and the control section students completed the same textbook homework, so these protocols were collected and analyzed for individual initiates of reflective abstraction.

These initiates provide evidence that the experimental curriculum was successful in initiating reflective abstraction. They also provide evidence that the teacher in the control section regularly initiated reflective abstraction in his lectures. Finally, these initiates seem to indicate that many calculus students engage in reflective abstraction regardless of the teacher or the type of curriculum.
CHAPTER 5
DISCUSSION

This chapter will examine the extent to which the experimental curriculum initiated reflective abstraction. It will also examine the extent to which the control section initiated reflective abstraction. Next it will compare the performance of the students in the two curricula. Finally, it will answer the question of whether or not reflective abstraction is the cause for improved performance on the concept of limit.

Theoretical Framework

Selden, Mason, and Selden (1989) have demonstrated that calculus students do not understand the fundamental concepts. These students are unable to solve nonroutine problems. Their research also suggests that calculus students recall very little calculus in later classes (Selden, Selden, Hauk & Mason, 1999). In order to improve understanding in calculus, one can examine the work of Jean Piaget (Beth & Piaget, 1966). He discusses reflective abstraction as a key to developing conceptual understanding. Piaget discusses interiorization, coordination, encapsulation, and generalization as the constructs of reflective abstraction. Dubinsky (1991) clarifies Piaget's definitions and he refines Piaget's notion of reversibility (Piaget, Inhelder & Szeminska, 1960) into a fifth construct, reversal.
These five categories seemed to fit well with the calculus curriculum, so this theoretical framework was chosen to help improve student performance in calculus.

The concept of limit was chosen because it is the first idea in calculus that is substantially different from algebra. This idea is central to the notion of derivative and integral that appear later in the curriculum. It seems reasonable to argue that a complete understanding of the concept of limit will greatly improve the chances of success in calculus. Therefore using the constructs of reflective abstraction to design a curriculum to improve student understanding of the concept of limit may be a key component to improving calculus teaching and learning.

The first question to answer is, “Can a curriculum that initiates reflective abstraction improve student performance on the concept of limit?” In order to answer this question one should ask, “Did the experimental curriculum promote reflective abstraction?”

Initiates

Instructor Initiates of Reflective Abstraction in the Experimental Curriculum

Through examination of field notes from class observations and audiotapes from class observation and conversations with the teacher, several examples of instructor initiates of reflective abstraction were identified.

The instructor in the experimental section was not informed about the design of the experiment. She was simply asked to implement the curriculum and to assist students while they were working in small groups. She was also told to clarify the
more challenging elements of the curriculum. While implementing the curriculum she regularly asked clarifying questions. These questions can be categorized as coordination, generalization, reversal, and encapsulation initiates.

**Interiorization**

The instructor initiated interiorization with a brief lecture on the algebraic procedure evaluating limits. During this lecture she focused on the procedural elements such as adding rational expressions, rationalizing the denominator and substitution. After her lecture she asked the students to complete a few of these problems. She circulated through the room and checked progress. After the students finished, she asked them to describe how they solved the problems. By assisting students with procedural understanding, the instructor was promoting interiorization.

While assisting students with the group work, she asked several questions to initiate reflective abstraction. She asked large numbers of coordination question and large numbers of generalization questions. A sample of the questions follows.

**Coordination**

1. If a function is heading to positive infinity, does the limit exist?
2. If the graph is connected and the table tells us that the limit exists, how does this relate to the definition of continuity?
3. How is the table related to the graph for a function with an asymptote?
4. How is division by zero related to the notion of an asymptote?

5. If you “have a zero in the denominator” after plugging in the value, must there be an asymptote?

6. How is the notion of removable discontinuity related to the division by zero issue?

7. How do functions $f$ and $g$ differ? (They agree at all but one point.) Should they have the same limit?

**Generalization**

1. How can one decide that a piecewise function is continuous?

2. How can a table be misleading? Can a table make it look like a limit exists but it really doesn’t?

3. How is this problem similar or different from the previous problem?

4. In light of what you now know, can you go back and solve the previous problem using a different strategy?

At the conclusion of the lessons she would address the entire class and ask clarifying questions. These questions were often of the reversal or encapsulation nature. A sample of these questions follows.

**Reversal**

1. If the function does not exist at a point, what does the graph look like?

2. If a function is not continuous, can a limit exist?

3. If a limit does not exist, what would the table look like?
Encapsulation

1. After looking at the formal definition of limit, what does it mean to you?
2. What does it really mean for a function to be continuous?

Despite no knowledge of the design of the experiment, the instructor successfully initiated interiorization, coordination, generalization, reversal and encapsulation. She did this by clarifying procedures and establishing relationships among concepts. She was comfortable with initiating interiorization and coordination, but the relatively small number of questions in the other areas indicates that she was less comfortable initiating generalization, reversal and encapsulation.

Peer Initiates of Reflective Abstraction in the Experimental Curriculum

Through examination of field notes from class observations and audiotapes from collaborative group work, several examples of peer initiates of reflective abstraction were inferred. The examples that follow are a sample of questions that students asked while working in collaborative groups. The questions indicate initiates of interiorization, coordination, generalization, reversal and encapsulation.

Interiorization

There were several examples of students executing procedures. The working student would explain how she was solving the problem. When other
students were unclear, they would ask questions about how a procedure is executed. Interiorization inferences follow.

Example 1: Use an algebraic procedure.
- How do I do this one?
- Remember how I did the homework with the opposite reciprocal? That is how to do it for Alice.

Example 2: Use a tabular procedure to find the limit.
- How do you solve this with the table?
- Plug in the numbers to find out what the values are.
- How does that help us find the limit?
- Where are the numbers going? What are they getting closer and closer to?

Coordination

The largest number of peer initiates was for the coordination category. Students tried to understand the various concepts and they wanted to see how these ideas were related. A collection of student coordination discussions follows.

Example 1: Coordinate graphic and algebraic representations.
- There is no fraction, so can there be a hole?
- There is nothing to cancel out so there is no hole.

Example 2: Coordinate several strategies to find a solution.
- What should I write for Carla?
• She looks at all three strategies.

• If you get the same answer for each, you have three ways to verify that your answer is correct.

**Example 3**: Coordinate several strategies to find a solution.

• Wait a minute, should there be a hole there?

• Yes.

• But why can’t we see it on the calculator?

• Because of how the $x$-axis is defined.

• Should Carla invalidate George’s method because we didn’t even see the hole in the graph? So the graph method might not be that good for this one.

• But it still would have a limit though.

• It is the same thing no matter what way you do it.

• But for convenience factor, George’s would not be the most convenient for this set up and Tom’s, it doesn’t exactly say .5 but you can infer from the information that the limit will approach .5.

• Because there is the same change every time.

• So you probably want to use Alice’s first and Tom’s to verify.

• Yeah.

• I think you should always try algebraically first off.

**Example 4**: Coordinate several strategies to find a solution.

• Alice gets 0/0, so she would say there is no limit.

• Next is Tom and he gets . . .
• Now George, zoom it out so we can see it.

• Carla would do the factoring. . . . (Students monitor the algebra.)

Generalization

There were relatively few examples of peer initiates of generalization.

Example 1: Extending a previous strategy to a new problem.

• So the same thing would have been applicable to what we just did?

• Yeah.

Example 2: Construct a graph with certain characteristics.

• Construct a graph.

• We are basically doing the opposite of the last one.

• How do we do that? Just put a negative sign in front?

• I have the graph.

• Good job.

• Construct a graph . . .

• Is there some way that we can manipulate this to make it as given?

• I don’t know. I forget how to make equations like that. . . .

• No. It didn’t work.

• Try one over one minus x.

• No. There needs to be a zero.

• That’s hard.

• You have to cube it. It is \( (x + 2) \) over \( (x - 1) \). Cool.
No examples of reversal or encapsulation were inferred among peer initiates in the experimental curriculum. The lack of student initiates in these areas may be related to the small number of instructor initiates in these areas.

Curriculum Initiates of Reflective Abstraction in the Experimental Curriculum

The curriculum was designed to initiate interiorization, coordination, generalization, reversal, and encapsulation with respect to the notion of limit. There were a large number of questions in most categories. A sample of these questions together with student answers follows.

Interiorization

Many questions in the curriculum ask students to complete procedures. These include constructing tables to suggest values of a limit, evaluating a limit algebraically, and determining whether or not a function is continuous at a given point. A sample of interiorization initiates together with student responses follows.

Example 1: Use a graphical procedure to find a limit.

- Question: Carla asks George to explain how the graph shows the function approaching the same value as $x$ approaches 2 from the left and the right.
  
  How will George answer Carla's question?

- Answer: The graph shows the line from the left and the right approaching $x = 2$ and the $y$-value is approaching $y = 2$.

Example 2: Use an algebraic procedure to find a limit.
• Question: Alice likes algebraic simplification. She claims in this case it is appropriate to plug 2 into \( g(x) \) in order to determine the behavior of the function as \( x \) approaches 2 from the left and from the right. How would she answer the question? Demonstrate the strategy.

\[
\frac{x^2 - 4x - 5}{x + 1}, \quad \frac{2^2 - 4 \cdot 2 - 5}{3} = \frac{4 - 8 - 5}{3} = \frac{-9}{3} = -3.
\]

Coordination

Many questions were asked to initiate coordination. The curriculum was designed to help students understand limits from a tabular, graphical and algebraic perspective. The students also were expected to understand how notions like asymptotes, removable discontinuities, and one-sided limits were related to the notion of limit. For these reasons the largest number of questions related to initiating coordination. A sample of curriculum questions and student answers follows.

Example 1: Coordinate notions of limit and continuity.

• Question: Will a function always approach the same number from both the left and the right? Write a paragraph. Include examples and counterexamples in your discussion. Discuss how this idea is related to other ideas in the unit.

• Answer: A function will not always approach the same number from both the left and the right. Cases where a function approaches the same number from the left and from the right include functions that are always continuous.
There are functions that are not continuous but still approach the same
number from the left and the right. In the graphs of this type of function
there is a hole.

Example 2: Coordinate division by zero and concept of asymptote.

- Question: Alice claims that you do not need to look at the graph of
  \[ f(x) = \frac{6}{x + 1} \]
  to know that there is an asymptote. Explain Alice's reasoning.

- Answer: Plugging \(-1\) into the equation gives a 0 in the denominator;
  therefore, the value does not exist and there is an asymptote.

Example 3: Coordinate plug-in rule and division by zero.

- Question: Carla claims that Alice's plug-in-the-value strategy fails here.
  Why does Carla make the claim that one cannot plug \(-1\) into the function?

- Answer: The plug-in-the-value strategy fails because \(-1\) gives a value of
  zero in the denominator.

Example 4: Coordinate algebraic solution and table solution.

- Question: Let \( f(x) = \frac{x^2 - 4}{x - 2} \). Evaluate \( \lim_{x \to 2} f(x) \).

- Answer: Student solves the problem two ways. The first uses the algebraic
  strategy recognizing that the function \( x + 2 \) agrees with the original function
  at all but one point, and then she substitutes in 2 to get an answer of 4. She
  also constructs an \( x, y \) table with \( x \)-values of 1.997, 1.998, 1.999, 2, 2.001,
  2.002, 2.003, and even though the function is not defined at 2, she concludes
  the limit is 4.
Example 5: Coordinate graphic and algebraic definitions of continuity.

- Question: Tom asks whether or not the function is continuous at $x = -3$.
  
  George likes graphs. How would George answer Tom's question?
  
  - Answer: The graph stops and continues at another place.
  
  - Question: Alice likes definitions. How would Alice answer Tom's questions?
  
  - Answer: The function is not continuous at $x = -3$ because plugging the value into the equation does not give a $y$-value.

Example 6: Coordinate the notions of limit and continuity.

- Question: Explain the relationship between the concept of a limit and the notion of a continuous graph.
  
  - Answer: The limit and the function of the same number must correspond for the notion of a continuous graph to exist. Therefore it can be stated that if the limit is equal to the function of that same number then it can be assumed that the function is continuous at that point.

Generalization

A few questions in the experimental curriculum were designed to help students generalize their ideas into other areas. Many of these questions asked students to hypothesize rules. Other questions asked students to construct functions.
with certain characteristics. There were relatively few generalization questions.

A sample of these generalization initiates together with student responses follows.

**Example 1:** Construct a graph with specific characteristics.

- **Question:** The instructor asks the team to construct a function \( q(x) \) and its graph such that both of the following statements are true: As \( x \) gets closer and closer to 3 from the left, \( q(x) \) gets increasingly positive without bound. As \( x \) gets closer and closer to 3 from the right, \( q(x) \) gets increasingly positive without bound.

- **Answer:** Student constructs the rule \( q(x) = \frac{4}{(x-3)^2} \) and she constructs the graph of the function.

**Example 2:** Extending the “division by zero” case to permit a limit.

- **Question:** Explain whether or not it is possible for a limit to exist if a “zero in the denominator” results after plugging in the appropriate value.

- **Answer:** There is a possibility for it to exist if plugging in the appropriate value also causes a zero in the numerator of the function.

**Example 3:** Construct a graph with specific characteristics.

- **Question:** Construct a function such that limit as \( x \) approaches 2 from the left is 1 but as it approaches 2 from the right it is 3.

- **Answer:** \( g(x) = \begin{cases} -(x-3) & \text{if } x > 2 \\ -(x-5) & \text{if } x \leq 2 \end{cases} \)
Encapsulation

There was one question in the curriculum that asked students to encapsulate their understanding of the concept of limit.

Example: Encapsulate the notion of limit.

- Question: Carla decides to write a summary of this collection of limit lessons in her notebook. She wants to write a definition in her own words and she wants to include relevant examples and counterexamples in her notes. Help Carla complete her task.

- Answer: A limit is when, as the function is getting closer and closer to the same $x$-value from the left and the right, the function is getting closer and closer to the same $y$-value from the left and from the right. Continuous graphs always have a limit for any $x$-value. You can draw a continuous graph without lifting your pencil from the paper. For a graph that is not continuous, you have to lift your pencil from the paper to keep drawing it. Functions that have a limit even though they are not continuous reach the same number from the left and from the right even if there may or may not be a $y$-value. Examples include functions with removable discontinuities that create holes in graphs. You can use the all-but-one-point rule to find the limit. A function that is not continuous and has no limit is because the function gets closer to a different number from the left and from the right. You can use tables or graphs to see this.
The curriculum initiated reversal by asking students to reverse definitions or to construct counterexamples. A sample of reversal questions and student definitions follows:

**Example 1:** Reverse the definition of the limit.
- **Question:** Construct a graph to show when a limit does not exist.
- **Answer:** If there is an asymptote at \( x = -1 \), then the limit does not exist at \( x = -1 \).

**Example 2:** Reverse the definition of a limit.
- **Question:** Construct a rule for when a limit does not exist.
- **Answer:** If the limit from the left does not equal the limit from the right, then the limit does not exist.

**Example 3:** Reverse the definition of a limit.
- **Question:** Will a function always approach the same number from both the left and the right? Write a paragraph. Include examples and counterexamples in your discussion. Discuss how this idea is related to other ideas in the unit.
- **Answer:** There are functions that do not follow this rule. This is when there is an asymptote.

**Individual Initiates of Reflective Abstraction in the Experimental Curriculum**

Students were assigned textbook questions for homework. This work was analyzed to find evidence of interiorization, coordination, generalization,
encapsulation and reversal. Since these examples of reflective abstraction occurred while students completed homework, they are classified as individual initiates.

**Interiorization**

Students regularly performed the procedures needed to solve the problems. They evaluated many limits using algebraic procedures. Procedural knowledge is a standard requirement in most textbook exercise sets, so interiorization was the most prevalent category in the homework.

**Coordination**

Students engaged in coordination on the textbook homework. Students would often use more than one strategy to evaluate the limits. They would use algebra, tables and graphs. The students also successfully coordinated notions of one-sided limits, limits, asymptotes and continuity. The following are a few examples of coordination from the homework.

**Example 1:** Coordinate limit and continuity.
- The limit exists and the function value exists, but they are not equal. The graph is not continuous, and there is a hole.

**Example 2:** Coordinate removable discontinuity and the all-but-one-point rule.
• The limit at $f(5)$ is $1/10$. Substitution doesn't work initially. Once the hole is “removed,” we can find the limit of the function based on the all-but-one-point rule.

**Example 3:** Coordinate limit and continuity.

• The value of the function at $c$ doesn’t match the limit of the graph at $c$, so the function is not continuous.

**Generalization**

There were few opportunities to demonstrate generalization of the limit concept. One such question was, “Is $\lim_{x \to 0} \sqrt{x} = 0$ true?” This type of example was never studied in class, so students had to decide how to extend the definition of the limit to this case. Some students recognized that it was impossible to approach 0 from the left, so they said as $x$ approaches 0, $f(x)$ tends to 0, so the limit exists. Others said it was false because it was impossible to approach 0 from the left. Many either skipped the problem or simply wrote true or false with no explanation.

**Encapsulation**

There were no opportunities in the traditional homework for students to demonstrate encapsulation.

**Reversal**
For certain true false questions, students needed to demonstrate why a statement was false. For example students had to reverse the limit definition to show why $\lim_{x \rightarrow 0} \frac{|x|}{x} = 1$ is a false statement. This demonstrates that the reversal was initiated in the control curriculum.

The evidence implies that the experimental curriculum was successful in initiating reflective abstraction through instructor, peer, curricular and individual initiates. Evidence of interiorization and coordination was seen most often. This was especially true with the peer initiates and the instructor initiate. Evidence for generalization and encapsulation was rare or nonexistent in the collaborative group work and the standard textbook questions. In order to increase opportunities for reflective abstraction, teachers should be encouraged to initiate generalization, reversal and encapsulation. Textbook problems may also be augmented with additional questions to initiate each of the five categories. Students in collaborative groups should be trained to ask more generalization, reversal and encapsulation questions. These recommendations would enable students to have more diverse opportunities to engage in reflective abstraction.

Initiates of Reflective Abstraction in the Control Curriculum
Interiorization

The instructor in the control section was given a traditional curriculum and he was asked to implement it. The instructor in this section did promote reflective abstraction through his lecturing style. He taught a traditional lecture using algebraic and graphical strategies for finding limits. By helping the students learn these procedures, he initiated interiorization.

Coordination

He regularly introduced new topics to his students by asking questions. Before he would lecture, he would always ask questions to help students connect the new ideas to previous ideas. These questions were designed to initiate coordination.

Example 1: Coordinate the rule of a function and continuity.

- Is it possible to make this function seamless? Is it possible to define it so well so that this is continuous?

Example 2: Coordinate all-but-one point rule, removable and nonremovable discontinuities.

- Is it possible to define that function so this would be called a nonremovable discontinuity and this would be called a removable discontinuity?

Example 3: Coordinating piecewise functions and continuity.
Can you do anything at \( x = 2 \) to define another function that would force it to be continuous?

Sometimes he would have to answer his own questions. Other times he would get into a discussion with a student or students. The following discussions occurred in his continuity lecture:

**Example 4: Coordinate limits and continuity.**

- **Teacher:** Does this limit exist? How would you approach this?
- **Student:** Substitute into the function.
- **Teacher:** Which part?
- **Student:** The one where it is less than.
- **Teacher:** You are talking sides now. What name does it have when \( x \) is approaching 2 from the left?
- **Student:** The top one.
- **Teacher:** Now we can use our substitution rule . . . What can you say about the first condition that has to be met?
- **Student:** It has to exist.
- **Teacher:** It does not exist, right?
- **Student:** Check the next one.
- **Teacher:** I need all three to work and the first fails, so is the function continuous at 2?
- **Student:** No.
Teacher: I don’t need to check the others since I need all three to be true. So therefore the function is not continuous at $x$ equals 2. Is it defined at 2?

What is it? Is it removable? Can we define it? Do we have a choice?

Student: It is defined.

Teacher: Is it discontinuous anywhere else?

Student: No.

Teacher: Forgetting about the restriction, this is a straight line and it is continuous everywhere.

**Example 5:** Coordinate informal and formal definitions of continuity.

Teacher: Can you draw this curve without picking up the chalk?

Student: No.

Teacher: And there is no place I could put this dot where I could make this happen. I have to pick up my chalk. One dot cannot fix the problem. On this problem is there anywhere I could put it? Yeah, right here. So in this case it is a . . . ?

Student: Nonremovable discontinuity.

**Example 6:** Coordinate limits and continuity with piecewise functions.

Teacher: Graphically, what are these two pieces?

Student: Lines.

Teacher: So what are you doing by changing the $a$?

Student: Changing the slope.
Teacher: So let's work this. What happens as $x$ approaches 1 from the left? What do you think that one is? What is the limit as you approach 1 from the left?

Student: $a + 3$.

Teacher: Very good, $a + 3$. So from the left this is $a + 3$. Why do we care about what it is from the right?

Student: They need to match.

Teacher: We want the limit to exist. So what is that equal to from the right?

Teacher: What has to be true? They have to be equal.

Student: That should be 5.

Teacher: Thank you. Are we done?

Student: We need the other two.

Teacher: What is this . . . ? What is that limit? Is there a hole? Two of the conditions are satisfied.

Student: You need the third.

Teacher: Is this equal to that?

**Generalization**

This instructor naturally promoted interiorization and coordination as part of his lecturing style. There were few occurrences of generalization. One example was asking students to construct a function with certain characteristics.
• Construct the graph of a function $g(x)$ such that as $x$ gets closer and closer to 4 from below, $g(x)$ gets closer and closer to 2, and as $x$ gets closer and closer to 4 from above, $g(x)$ gets closer and closer to 2. Also, let $g(4) = 2$.

Reversal

There were not many examples of reversal in this curriculum. One example was a question when he asked students to reverse the definition of continuity to explain why a function was not continuous.

• Let $g(x) = \frac{x^2 - 4}{x + 2}$. Is $g(x)$ continuous at $x = -2$?

After asking the question to the students, the instructor proceeded to explain why this function was not continuous. He also discussed the possibly misleading graph of this function on the graphing calculator.

Encapsulation

This instructor initiated encapsulation by sharing a metaphor for the concept of limit. He referred to this as the “hook” method. He told the students to imagine that they are walking along the $x$-axis and they are holding a hook that is attached to the graph like a clothesline. He told the students to imagine that as you walked closer and closer to $x = 2$ on the number line, what does the hook get closer and closer to? Students regularly referred to his hook metaphor when they asked him questions about the limit. When asked what the limit concept really means, several
of his students discussed the hook metaphor in the interviews. This metaphor helped students begin to encapsulate the concept of limit. He also helped students encapsulate the notion of piecewise function by comparing it to names. He said your friends call you one name and your family calls you a different name. So your name depends on where you are. He said this is also true with piecewise functions.

After completing the lecture series, the instructor was asked if he regularly teaches by asking questions. He responded that he has found that this is the best way to get students to pay attention in class and it also seems to help his students learn. He believes that many of his students do not benefit from a typical lecture. When asked whether or not he regularly constructs metaphors like the “hook” for his students, he responded that he tries to construct ideas that will make the mathematics seem more concrete and less abstract. He believes that this also helps students better understand the mathematical concepts.

The instructor in the control curriculum did promote reflective abstraction in his lectures. His natural teaching style has evolved to promote coordination through questioning and to help students begin to encapsulate through construction of metaphors. Clearly he is capable of promoting reflective abstraction through instructor initiates.

The instructor did not encourage students to work together in the classroom, so no peer initiates were inferred. Most of the control students did little reflection on the standard homework, so there were no opportunities to observe individual initiates of reflective abstraction. Finally, the curriculum was not designed to
promote reflective abstraction, so there were no opportunities to infer curricular
initiates.

A Reflection on the Study and Its Components

The sources of data for this study included a pretest on the concept of limit, a
posttest on the concept of limit, transcribed audiotapes of interviews with students
clarifying their understanding on the posttest, transcribed audiotapes from a sample
of lessons, transcribed audiotapes from a sample of collaborative group work, field
notes from class observation, informal discussions with the teachers and completed
student in-class projects and homework projects. The pretest and posttest
instruments were designed to reflect typical questions regarding the limit concept
from the calculus curriculum. These questions were refined after consultation with
a university calculus instructor and a two-year college calculus instructor. The
pretest and the posttest were designed so that the questions were in parallel, that is
each of the questions on the pretest corresponded to a similar question on the
posttest. Two additional writing questions were included on the posttest.

The pretests were scored using a two-point rubric. Students earned one
point for a correct solution and zero points for an incorrect solution. This is very
important because the students self-selected their calculus section.

The pretests and posttests were scored using a three-point short-response
rubric from the Illinois State Board of Education (2005b). Students earned two
points for a correct solution, one point for a partially correct solution and zero points
for an incorrect solution. The researcher and the independent grader developed
item-specific scoring schemes so that scoring was consistent. The analysis of the pretests found no significant difference between the experimental group and the control group. The analysis of the posttests showed a significant statistical difference with the students in the experimental section outperforming the students in the control section.

To get additional information, a comparison subgroup was chosen from the students in the study. These students were chosen based on their level of improvement from the pretest to the posttest. Two students with greatest improvement, two students with median-level improvement, and two students with least improvement were chosen from each section. These twelve students had their posttests rescored using the five-point extended response rubric from the Illinois State Board of Education (2005a). The tests were scored three times, first for mathematical knowledge, second for strategic knowledge and third for explanation. No significant difference was identified for mathematical knowledge or strategic knowledge. This is understandable because the sample sizes were so small. A significant difference favoring the experimental section was identified on the communication measure. The students in the experimental section were given many opportunities to write and the students in the control section were not given such opportunities. The assignments were designed to promote reflective abstraction, so such writing assignments are a contributing factor in improving a student’s written communication of the limit concept.

The comparison subgroup was analyzed to determine the characteristics of effective students. Three subgroups were constructed from the original twelve
students. Six students who scored below the median on the pretest and above
the median on the posttest were classified as the Improve subgroup. Two students
who scored above the median on the pretest and above the median on the posttest
were classified as the Maintain subgroup. Four students who score above the
median on the pretest but below the median on the posttest were classified as the
Regress subgroup.

These students were interviewed to clarify their understanding of the concept
of limit. These interviews were audiotaped and transcribed. The transcripts were
analyzed to infer various constructs of reflective abstraction. Subcategories of
interiorization, coordination, encapsulation, generalization, and reversal were
constructed as an aid in coding. Inferences from the evidence suggest that the
students who improved engaged in reflective abstraction more often and the
students who regressed engaged in reflective abstraction less often. The total
number of inferences of the constructs of reflective abstraction by group appears in
Table 12.
Among the constructs of reflective abstraction, coordination was inferred most often. Students in each of the three subgroups had several inferences of coordination. This suggests that coordination is necessary to begin to develop conceptual understanding of the limit. Students in the three subgroups had varying numbers of inferences of generalizations. The six students in the Improve group had an average of 3.33 occurrences of generalization. The two students in the Maintain group had an average of 1.00 occurrence of generalization and the four students in the Regress group had an average of 1.00 occurrence of generalization. This indicates that generalization may be a key to developing understanding of the concept of limit.

As was expected, the instructor in the experimental section initiated reflective abstraction several times. However, it was unexpected when the instructor in the control curriculum also initiated reflective abstraction. He regularly initiated coordination through his questions and he constructed metaphors that

---

**Table 12**

<table>
<thead>
<tr>
<th>Group</th>
<th>Interiorization</th>
<th>Coordination</th>
<th>Generalization</th>
<th>Reversal</th>
<th>Encapsulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Improve</td>
<td>40</td>
<td>63</td>
<td>20</td>
<td>23</td>
<td>2</td>
</tr>
<tr>
<td>Maintain</td>
<td>9</td>
<td>30</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Regress</td>
<td>19</td>
<td>36</td>
<td>4</td>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

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helped his students construct personal meaning. Many of his students were successful with the limit concept and it is reasonable to suggest that his students were successful because he initiated reflective abstraction.

The students in the experimental section regularly used collaborative groups and those in the control section did not. The collaborative group work was designed to promote reflective abstraction through peer initiates. Therefore, collaborative group work designed to promote reflective abstraction is a contributing factor in improving student performance on the concept of limit.

Both sections were required to cover the same material. The examples used in the experimental section were identical to those used in the control curriculum. Both instructors taught using Calculus by Larson, Hostetler and Edwards (2006) and they were to cover material in sections 1.2, 1.3, and 1.4. All students were given the same homework assignments from Larson et al. (2006). The students in the experimental section were more likely to write extended answers on the textbook homework. This is not surprising because the curriculum provided many writing opportunities for students. In general the students with greatest improvement seemed to take their homework more seriously than students with the least improvement. This may be due to the fact that the better students had more individual initiates of reflective abstraction than the weaker students.

Both sections regularly used technology. The students in the experimental curriculum were required to have a graphing calculator to complete the assignments. The instructor in the experimental section guided their use of the calculator. The instructor in the control curriculum used a calculator in his lectures. He also taught
the students how to use the calculator. Calculators were not permitted on the pretest or the posttest, so this was not a contributing factor.

The curriculum can be seen as an aid in promoting reflective abstraction but it is not sufficient. Several students in the experimental section did not perform well. They were given several opportunities to engage in reflective abstraction. Yet many ignored assignments or provided terse responses. A few of these students indicated displeasure with the project. They did not think that they should be required to write about mathematics or explain their reasoning. Their negative attitudes toward the project together with their lack of interest in reflective abstraction may have been contributing factors to their lack of success.

Limitations

One limitation with this study was the sample sizes of $n_1 = 16$ and $n_2 = 19$. A second limitation was the five-day instructional period. A longitudinal study with a much larger sample size would help address these concerns. However, despite the small sample sizes and the one-week time frame, positive effects were identified.

Many studies would randomize assignment of students and teachers to the two sections. This was not possible given the usual enrollment patterns of students and scheduling patterns of teachers. The pretest-posttest model addresses this concern. Since no significant difference was identified on the pretest, it would seem fair to compare performances on the posttest.

The experiment was designed to measure the effect of the curriculum on student performance of the concept of limit. Many studies discuss the role of
teacher effect on student performance. Nye, Konstantopoulos and Hedges (2004) demonstrate a strong teacher effect with elementary school mathematics students. Guskey and Easton (1983) show that teacher effect is the most significant factor in student achievement for community college students.

In an attempt to control for teacher effect, two teachers with similar experience participated in the study. Each had six years experience teaching in higher education and six years experience teaching calculus. The teacher in the control curriculum had a terminal degree in mathematics and the teacher in the experimental curriculum had a Master of Arts degree in mathematics. In comparing the two teachers it was identified that the teacher in the control section promoted reflective abstraction as part of his natural teaching style. He used questioning and metaphors to initiate reflective abstraction. Although it was not part of the study, the data from the classroom observation suggest that if there were a teacher effect, it would have been be in favor of the students in the control section.

Directions for Future Study

This study examined whether or not a curriculum that promotes reflective abstraction can improve student performance on the concept of limit. The students in this study were enrolled in calculus at a community college. Researchers may determine if the findings are similar for high school AP calculus students and university calculus students.
The concept of the limit is the first calculus concept studied and it is a foundation for most of the other concepts in calculus. Cornu (1991) writes, "The mathematical concept of a limit is a particularly difficult notion, typical of the kind of thought required in advanced mathematics. It holds a central position, which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity, and of differential and integral calculus" (p. 153). Improving understanding of the limit concept may help students throughout their studies in calculus. Researchers may ask whether or not a curriculum designed to promote reflective abstraction extended to all calculus topics could improve student performance in calculus.

The reflective abstraction model may also be helpful in K-12 mathematics. There are certainly aspects of interiorization, coordination, encapsulation, generalization and reversal in arithmetic, algebra, and geometry. Curricula that promote reflective abstraction may improve student performance in K-12 mathematics.

Research indicates that teacher effect is very pronounced in mathematics. One might ask, "What are the traits of an effective teacher?" It may be beneficial to study effective and less effective teachers to determine to what degree the teachers promote reflective abstraction.
Conclusion

Research Question: Can a curriculum that initiates reflective abstraction improve student performance on the concept of limit? The following evidence suggests that reflective abstraction is a contributing factor for improved student understanding of the limit concept.

The evidence suggests that the experimental curriculum was successful in promoting reflective abstraction through individual, peer, curricular, and instructor initiates. The control curriculum was not designed to promote reflective abstraction. However, as one might expect from a good instructor, the control curriculum promoted reflective abstraction through instructor initiates. The students in the experimental section outperformed the students in the control section on a test of the concept of limit. Both sections examined similar examples in class and completed the same homework exercises. For these reasons it is fair to conclude that the curriculum was a significant reason for the success of the students in the experimental section.

An interesting unsuspected result was that students in the experimental section were better at written communication of mathematics than were the students in the control section. This indicates that opportunities to reflect on learning, together with regular writing assignments, may improve a student's written communication skills in mathematics.

Analysis of the data from the comparison subgroup suggests that students with the greatest improvement engage in reflective abstraction more often than students with less improvement. Therefore one may argue that reflective abstraction
is significant factor in student performance on the concept of limit. Further examination of this data shows that generalization may be the key to developing understanding of the concept of limit.

This study demonstrates that a calculus curriculum can promote reflective abstraction. Furthermore, such a curriculum together with instructor, peer, curriculum and individual initiates improves student performance and written communication on the concept of limit.

Reflective abstraction is an effective tool for improving a student’s performance in mathematics. The constructs of interiorization, coordination, encapsulation, generalization and reversal should be examined in the process of mathematics curriculum development. Teachers should promote reflective abstraction through instructor, peer, and curricular initiates. They should design problem sets that enable students to initiate reflective abstraction independently. The challenges of teaching and learning mathematics are substantial. Promoting reflective abstraction will enable teachers to help students meet this challenge.
REFERENCES


1. Find the limit of $f(x)$ as $x \to 0$. Carefully explain your reasoning.

2. Find the limit of $f(x)$ as $x \to 2$. Carefully explain your reasoning.
3. Find the limit of \( f(x) \) as \( x \to 1 \). Carefully explain your reasoning.

4. Find the limit of \( f(x) \) as \( x \to 0 \). Carefully explain your reasoning.
5 - 8. Calculate each of the following limits, if the limit exists. If a limit does not exist, explain why it does not.

5. Evaluate \( \lim_{x \to 3} \frac{x - 2}{x^2 + 1} \). Carefully explain your reasoning.

6. Evaluate \( \lim_{x \to 2} \frac{1}{(x - 2)^2} \). Carefully explain your reasoning.

7. Evaluate \( \lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} \). Carefully explain your reasoning.
8. Evaluate \( \lim_{x \to 4} \frac{4-x}{2-\sqrt{x}} \). Carefully explain your reasoning.

9. Evaluate \( \lim_{x \to 3} f(x) \). Carefully explain your reasoning.
10. First class mail rates are listed in the following table.

First ounce $0.37  
Each additional ounce $0.23  

<table>
<thead>
<tr>
<th>Weight not over (ounces)</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1*</td>
<td>$0.37</td>
</tr>
<tr>
<td>2</td>
<td>0.60</td>
</tr>
<tr>
<td>3</td>
<td>0.83</td>
</tr>
<tr>
<td>4</td>
<td>1.06</td>
</tr>
<tr>
<td>5</td>
<td>1.29</td>
</tr>
</tbody>
</table>

a) If a letter weighs 0.7 ounces, what is the cost?

b) If a letter weighs 1.3 ounces, what is the cost

c) If \( x \) is the weight in ounces of a letter and \( C(x) \) is the cost to mail the letter determine \( \lim_{x \to 4} C(x) \). Carefully explain your reasoning.
11. Is there an $a$ such that $\lim_{x \to 3} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}$ exists? Explain your answer.

12. Determine a number $a$ such that the function $g$ defined by
$$g(x) = \begin{cases} x^2 - 2, & x \leq 3 \\ 2x + a, & x > 3 \end{cases}$$

is continuous on the entire real line.
APPENDIX B

POSTTEST INSTRUMENT
Name ________________________

1. Find the limit of $f(x)$ as $x \to 0$. Carefully explain your reasoning.

2. Find the limit of $f(x)$ as $x \to 2$. Carefully explain your reasoning.
3. Find the limit of $f(x)$ as $x \to 1$. Carefully explain your reasoning.

4. Find the limit of $f(x)$ as $x \to 3$. Carefully explain your reasoning.
5. Write one or more paragraphs on the significance or meaning of the following limit. You may use examples or definitions in your descriptions if you desire, but you must explain the limits in your own words. A definition copied from a book will not be accepted unless it is clearly explained in your own words. \( \lim_{x \to 3}(4x + 2) = 14 \)

6. Calculate each of the following limits, if the limit exists. If a limit does not exist, explain why it does not.

6. Evaluate \( \lim_{x \to 4} \frac{x - 3}{x^2 + 9} \) Carefully explain your reasoning.
7. Evaluate \( \lim_{{x \to 3}} \frac{-1}{(x - 3)^2} \) Carefully explain your reasoning.

8. Evaluate \( \lim_{{x \to 2}} \frac{x^2 - 6x + 8}{x - 2} \) Carefully explain your reasoning.
9. Evaluate \[ \lim_{{x \to 3}} \frac{9-x}{3-\sqrt{x}} \] Carefully explain your reasoning.

10. A friend of yours who recently enrolled in calculus is wondering what calculus is all about because he/she has heard you frequently use the word "limit." What short explanations, sentences, or examples would you use to explain to your friend what the "limit" is all about?
11. Evaluate $\lim_{x \to 3} f(x)$. Carefully explain your reasoning.
12. First class mail rates are listed in the following table.

<table>
<thead>
<tr>
<th>Weight not over (ounces)</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1*</td>
<td>$0.37</td>
</tr>
<tr>
<td>2</td>
<td>0.60</td>
</tr>
<tr>
<td>3</td>
<td>0.83</td>
</tr>
<tr>
<td>4</td>
<td>1.06</td>
</tr>
<tr>
<td>5</td>
<td>1.29</td>
</tr>
</tbody>
</table>

a) If a letter weighs 1.8 ounces, what is the cost?

b) If a letter weighs 2.2 ounces, what is the cost?

c) If \( x \) is the weight in ounces of a letter and \( C(x) \) is the cost to mail the letter determine \( \lim_{x \to 2} C(x) \)
13. Is there an $a$ such that $\lim_{x \to 3} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}$ exists? Explain your answer.

14. Determine a number $a$ such that the function $g$ defined by

$$g(x) = \begin{cases} 
 x^2 - 1, & x \leq 2 \\
 7x + a, & x > 2 
\end{cases}$$

is continuous on the entire real line.
Mathematical Knowledge
Knowledge of mathematical principles and concepts which result in a correct solution to a problem.

<table>
<thead>
<tr>
<th>Score</th>
<th>Description</th>
</tr>
</thead>
</table>
| 4     | *shows complete understanding of the problem's mathematical concepts and principles  
*uses appropriate mathematical terminology and notations including labeling answer if appropriate; (e.g. labels answers as appropriate)  
*executes algorithms completely and correctly |
| 3     | *shows nearly complete understanding of the problem’s mathematical concepts and principles  
*uses nearly correct mathematical terminology and notations  
*executes algorithms completely; computations are generally correct but may contain minor errors |
| 2     | *shows some understanding of the problem's mathematical concepts and principles  
*may contain major computational errors |
| 1     | *shows limited to no understanding of the problem’s mathematical concepts and principles  
*may misuse or fail to use mathematical terms  
*may contain major computational errors |
| 0     | *no answer attempted |
### Strategic Knowledge
Identification of important elements of the problem and the use of models, diagrams, symbols, and/or algorithms to systematically represent and integrate concepts.

<table>
<thead>
<tr>
<th>Score</th>
<th>Description</th>
</tr>
</thead>
</table>
| 4     | *identifies all the important elements of the problem and shows complete understanding of the relationships among elements  
*reflects an appropriate and systematic strategy for solving the problem  
*gives clear evidence of a complete and systematic solution process |
| 3     | *identifies most of the important elements of the problem and shows general understanding of the relationships among them  
*reflects an appropriate strategy for solving the problem  
*solution process is nearly complete |
| 2     | *identifies some important elements of the problem but shows only limited understanding of the relationships among them  
*appears to reflect an appropriate strategy, but the application of the strategy is unclear, or a related strategy is applied logically and consistently  
*gives some evidence of a solution process |
| 1     | *fails to identify important elements or places too much emphasis on unimportant elements  
*may reflect an inappropriate or inconsistent strategy for solving the problem  
*gives minimal evidence of a solution process; process may be difficult to identify  
*may attempt to use irrelevant outside information |
| 0     | *no apparent strategy |

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Explanation

Written explanation and rationales that translate into words the steps of the solution process and provide justification for each step. Though important, the length of response, grammar and syntax are not the critical elements of this dimension.

4
* gives a complete written explanation of the solution process employed; explanation addresses both what was done and why it was done
* may include a diagram with a complete explanation of all its elements

3
* gives a nearly complete written explanation of the solution process employed; clearly explains what was done and begins to address why it was done
* may include a diagram with most of the elements explained

2
* gives some written explanation of the solution process employed, either explains what was done or addresses why it was done; explanation is vague or difficult to interpret
* may include a diagram with some of the elements explained

1
* gives minimal written explanation of the solution process; may fail to explain what was done and why it was done
* explanation does not match presented solution process
* may include minimal discussion of elements in diagram; explanation of significant elements is unclear

0
* no written explanation of the solution process is provided
APPENDIX D

EXPERIMENTAL CURRICULUM
Limits
Hour 1

Goal: To understand the concept of a limit using tables, graphs and rules and to use the concept of a limit to understand the concept of continuity.

Scenario
Alice, Tom, Carla and George are working together as part of a study group. Their task is to complete the following lesson on limits.

Activity A
Let \( f(x) = x^2 + 2x + 1 \). Input this function into Y1 on the calculator.

Press \( \text{2nd} \) \( \text{WINDOW} \) (TBLSET) Change the values so that they match those below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.993</td>
<td></td>
</tr>
<tr>
<td>2.994</td>
<td></td>
</tr>
<tr>
<td>2.995</td>
<td></td>
</tr>
<tr>
<td>2.996</td>
<td></td>
</tr>
<tr>
<td>2.997</td>
<td></td>
</tr>
<tr>
<td>2.998</td>
<td></td>
</tr>
<tr>
<td>2.999</td>
<td></td>
</tr>
</tbody>
</table>
Question A2

When you choose x-values smaller than three, as the x-values get closer and closer to three what do the f(x) values get closer and closer to?

Tom's Answer:

Activity B

Press \( \boxed{\text{2nd}} \) WINDOW (TBLSET) Change the values so that they match those below.

<table>
<thead>
<tr>
<th>TABLE SETUP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{TblStart}=3.007 )</td>
</tr>
<tr>
<td>( \text{aTbl}=-0.01 )</td>
</tr>
<tr>
<td>( \text{Indpnt}: \text{Ask} )</td>
</tr>
<tr>
<td>( \text{Depend}: \text{Ask} )</td>
</tr>
</tbody>
</table>

Question B1

Tom is asked to use his calculator to find the values. Complete Tom's table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.007</td>
<td></td>
</tr>
<tr>
<td>3.006</td>
<td></td>
</tr>
<tr>
<td>3.005</td>
<td></td>
</tr>
<tr>
<td>3.004</td>
<td></td>
</tr>
<tr>
<td>3.003</td>
<td></td>
</tr>
<tr>
<td>3.002</td>
<td></td>
</tr>
<tr>
<td>3.001</td>
<td></td>
</tr>
</tbody>
</table>

Question B2

When you choose x-values larger than three, as the x-values get closer and closer to three what do the f(x) values get closer and closer to?

Tom's Answer:

The instructor refers to the previous two questions to define "approaching from the left" and "approaching from the right."
The instructor reminds the students
In Question A2 . . .
When you choose x-values smaller than three, as the x-values get closer and closer
to three what do the f(x) values get closer and closer to?

This question can also be asked using the following terminology.

As x approaches 3 from left, what does f(x) get closer and closer to?

In Question B2 we asked . . .
When you choose x-values larger than three, as the x-values get closer and closer to
three what do the f(x) values get closer and closer to?

This question can also be asked using the following terminology.

As x approaches 3 from the right, what does f(x) get closer and closer to?

Question B3
Recall \( f(x) = x^2 + 2x + 1 \)
Alice attempts to answer the following questions. Complete her task.

| As x gets closer and closer to 3 from the left, \( f(x) \) gets closer and closer to
| _______. |
| As x gets closer and closer to 3 from the right, \( f(x) \) gets closer and closer to
| _______. |

George claims that the table may not tell the entire story. He believes that as \( x \) gets
even closer to 3 that the y-values could change drastically. He thinks that a graph
is much better than a table to analyze a function's behavior near \( x = 3 \). He
constructs a graph of the function near 3.
Based on George's graph, answer the following questions.

As $x$ gets closer and closer to 3 from the left, $f(x)$ gets closer and closer to _____.

As $x$ gets closer and closer to 3 from the right, $f(x)$ gets closer and closer to _____.

Alice and Tom believe that the tables and the graphs must always match.

George believes that the table could provide incorrect evidence.

Carla listens to both arguments.

How will Carla respond to the question?

(Teacher will demonstrate how a table could be misleading.)
Place students in groups to complete this activity

Activity C

Let \( k(x) = \frac{6}{x+1} \)

Tom will find the values. Complete his task.

Use the \( Y= \) button to input the function as follows.

\[
\begin{align*}
Y_1 &= \frac{6}{x+1} \\
Y_2 &= \\
Y_3 &= \\
Y_4 &= \\
Y_5 &= \\
Y_6 &= \\
Y_7 &= 
\end{align*}
\]

Press \( \boxed{\text{2nd}} \) \( \boxed{TBLSET} \) and highlight \( \text{Ask} \) after \( \text{Indpnt}: \)

\[
\begin{array}{l}
\text{Depend:} \quad \text{Auto} \\
\text{as}\text{Kst}
\end{array}
\]

Press \( \boxed{\text{2nd}} \) \( \boxed{\text{TABLE}} \)

Input 1.9 under X. Press \( \boxed{\text{ENTER}} \)

\[
\begin{array}{|c|c|}
\hline
X & Y_1 \\
\hline
1.9 & 2.069 \\
\hline
\end{array}
\]

Input 1.99 under Y. Press \( \boxed{\text{ENTER}} \)

\[
\begin{array}{|c|c|}
\hline
X & Y_1 \\
\hline
1.99 & 2.0067 \\
\hline
\end{array}
\]

Continue this process and complete the table.
Question C1

<table>
<thead>
<tr>
<th>$x$</th>
<th>$k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td></td>
</tr>
<tr>
<td>1.99</td>
<td></td>
</tr>
<tr>
<td>1.999</td>
<td></td>
</tr>
<tr>
<td>1.9999</td>
<td></td>
</tr>
</tbody>
</table>

Question C2

Tom notices...
As $x$ gets closer and closer to 2 from the left, $k(x)$ gets closer and closer to _____.

Question C3

Tom completes the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td></td>
</tr>
<tr>
<td>2.01</td>
<td></td>
</tr>
<tr>
<td>2.001</td>
<td></td>
</tr>
<tr>
<td>2.0001</td>
<td></td>
</tr>
</tbody>
</table>

Question C4

As $x$ gets closer and closer to 2 from the right, $k(x)$ gets closer and closer to _____.

Question C5

Carla notices that $k(x)$ approaches the same number as $x$ approaches 2 both from the left and from the right?

She wonders if this will always be the case.
Reflective Question 1. (Answer this question after you have completed the unit)

Will a function always approach the same number from both the left and from the right?
Write a paragraph. Include examples and counterexamples in your discussion.
Discuss how this idea is related to other ideas in the unit.
George examines the graph

Carla asks George to explain how the graph shows the function approaching the same value as $x$ approaches 2 from the left and the right.

How will George answer Carla’s question?

Carla asks George to construct the graph of a function that approaches one number as $x$ approaches 2 from the left, yet it approaches a different number as $x$ approaches 2 from the right. If possible, help George construct such a graph.
Tom asks the group again to examine \( k(x) = \frac{6}{x+1} \)

Question C6
He constructs the following table. Complete his task.

Press \(2^{nd}\) TABLE and type over the values in the X column.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>2.069</td>
</tr>
<tr>
<td>1.99</td>
<td>2.006</td>
</tr>
<tr>
<td>1.999</td>
<td>2.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.1</td>
<td></td>
</tr>
<tr>
<td>-1.01</td>
<td></td>
</tr>
<tr>
<td>-1.001</td>
<td></td>
</tr>
<tr>
<td>-1.0001</td>
<td></td>
</tr>
</tbody>
</table>

Tom claims that as \(x\) gets closer and closer to -1 from below that \(k(x)\) gets closer and closer to -60000

Alice claims that there is no value that \(k(x)\) gets closer and closer to.

Question C7

Why does Tom think that \(k(x)\) gets closer and closer to \(-60000\) as \(x\) gets closer and closer to -1 from the left?

Question C8

Why does Alice believe that there is no value that \(k(x)\) gets closer and closer to?
Question C9
Recall \( k(x) = \frac{6}{x + 1} \)
Tom completes the following table. Help Tom complete the task.
Press 2nd TABLE and type over the values in the X column.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( k(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td></td>
</tr>
<tr>
<td>-0.99</td>
<td></td>
</tr>
<tr>
<td>-0.999</td>
<td></td>
</tr>
<tr>
<td>-0.9999</td>
<td></td>
</tr>
</tbody>
</table>

Question C10
Based on their answers to previous questions what do you think Tom and Alice will say to the question?

"What does \( k(x) \) get closer and closer to as \( x \) gets closer and closer to \(-1\) from the right?"

Tom: 
Alice: 

Carla recognizes that there was little debate about
\[ k(x) = \frac{6}{x + 1} \text{ as } x \text{ gets closer and closer to } 2 \]

but much debate about
\[ k(x) = \frac{6}{x + 1} \text{ as } x \text{ gets closer and closer to } -1. \]

She claims that something different happens at
\[ x = -1 \text{ for the function } k(x) = \frac{6}{x + 1}. \]

Question C10
Explain why Carla believes that something different is happening.
George again examines the graph.

George says that the graph has an asymptote at $x = -1$.

Alice claims that you do not need to look at the graph of $k(x) = \frac{6}{x + 1}$ to know that there is an asymptote at $x = -1$.

Explain Alice's reasoning.

**Question C11**

The instructor says in this case the limit does not exist. Help the students complete the following statement.

Alice and George claim that the limit cannot exist because of the graph.

The students decide to construct a rule for this case.

Help the students complete their task.

If

[Blank space]

then the limit does not exist at $x = -1$.

Each group of students will present their rule to the class as a whole. Then the instructor will lead a discussion to develop the rule for the class.
Reflective Question 2. (Answer this question after you have completed the unit) Write a paragraph. Include examples and counterexamples in your discussion. Discuss how this idea is related to other ideas in the unit.

1. Explain how the "division by zero" concept is related to the notion of a vertical asymptote.

2. Explain whether or not it is possible for a limit to exist if a "zero in the denominator" results after "plugging in" the appropriate value.

END HOUR 1
Limits
Hour 2

Activity D

The students examine the following function.

Let \( f(x) = \frac{x^2 - 5x + 6}{x - 2} \)

Alice recognizes that the numerator can be factored.

She writes

\[
  f(x) = \frac{x^2 - 5x + 6}{x - 2} = \frac{(x-3)(x-2)}{(x-2)}
\]

Tom decides to construct a table.

He inputs this function into Y1.

He changes the values of TBLSET to match those below.

He presses 2^{nd} TABLE
Question D1
Complete Tom's table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.993</td>
<td></td>
</tr>
<tr>
<td>1.994</td>
<td></td>
</tr>
<tr>
<td>1.994</td>
<td></td>
</tr>
<tr>
<td>1.996</td>
<td></td>
</tr>
<tr>
<td>1.997</td>
<td></td>
</tr>
<tr>
<td>1.998</td>
<td></td>
</tr>
<tr>
<td>1.999</td>
<td></td>
</tr>
</tbody>
</table>

Question D2

Tom notices as $x$ gets closer and closer to 2 from the left, $f(x)$ gets closer and closer to ______.

He changes the values of TBLSET to match those below.

```
TABLE SETUP
TblStart=2.007
Tbl=+.001
Indpnt: AUTO Ask
Depend: AUTO Ask
```

Question D3
He completes the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.007</td>
<td></td>
</tr>
<tr>
<td>2.006</td>
<td></td>
</tr>
<tr>
<td>2.005</td>
<td></td>
</tr>
<tr>
<td>2.004</td>
<td></td>
</tr>
<tr>
<td>2.003</td>
<td></td>
</tr>
<tr>
<td>2.002</td>
<td></td>
</tr>
<tr>
<td>2.001</td>
<td></td>
</tr>
</tbody>
</table>

Question D4

Tom notices as $x$ gets closer and closer to 2 from the right, $f(x)$ gets closer and closer to ______.
Question D5
Carla asks, "Does \( f(x) \) approach the \textbf{same number} as \( x \) approaches 2 both from \textbf{the left} and from \textbf{the right}?"

Answer:

Alice says that you do not need to check both "from the left" and "from the right". She believes if you know one of the answers the other has to be the same.

Tom, George and Carla are not sure.

They agree to answer the question (Reflective Question 1) when they have finished the unit.

Question D6
Recall \( f(x) = \frac{(x^2 - 5x + 6)}{(x - 2)} = \frac{(x - 3)(x - 2)}{(x - 2)}. \)

Alice says that it is \textbf{impossible} to determine what happens to \( f(x) \) as \( x \) gets closer and closer to 2 (either from the left or from the right) because \( f(2) \) \textbf{does not exist}.

Why does Alice believe that \( f(2) \) does not exist?

Recall that Tom likes using the tables on his calculator. He believes it is \textbf{possible} to determine what happens to \( f(x) \) as \( x \) gets closer and closer to 2.
Help Tom complete his tables.

\[ f(x) = \frac{(x^2 - 5x + 6)}{(x - 2)} \]

George decides to look at a graph.

Press ZOOM. Highlight 4: ZDecimal

Copy George’s graph.

How would Tom answer the following questions?

What happens to \( f(x) \) as \( x \) gets closer and closer to 2 from the left?

What happens to \( f(x) \) as \( x \) gets closer and closer to 2 from the right?
How would George answer the following questions?

What happens to $f(x)$ as $x$ gets closer and closer to 2 from the left?

What happens to $f(x)$ as $x$ gets closer and closer to 2 from the right?

Alice is surprised that there is no asymptote for $f(x) = \frac{x^2 - 5x + 6}{x - 2}$.

Carla explains that you do not need to look at the graph to know this. How does Carla know that there is no asymptote even though there is a zero in the denominator when you plug 2 into the function.

Explain Carla’s reasoning.
Question D9

After listening to Tom’s argument, Alice says that the function
\[ f(x) = \frac{(x^2 - 5x + 6)}{(x - 2)} = \frac{(x - 3)(x - 2)}{(x - 2)} \]
is exactly the same as the function \( g(x) = x - 3 \).

Tom believes the two functions are different because if \( x = 2 \) the word ERROR shows up in the table.

George believes that the two functions \( f(x) = \frac{(x^2 - 5x + 6)}{(x - 2)} = \frac{(x - 3)(x - 2)}{(x - 2)} \) and \( g(x) = x - 3 \) are a little different.

How do his graphs help him reach this conclusion?

Graph of \( f(x) \)   

Graph of \( g(x) \)

Explain the similarities and differences between the two graphs.
Carla has heard Alice’s concerns, she has looked at George’s graphs and she has looked at Tom’s tables. She decides to answer the questions.

Let \( f(x) = \frac{x^2 - 5x + 6}{x - 2} = \frac{(x-3)(x-2)}{x-2} \)

Carefully explain how she solved the problem.

1. Discuss the behavior of \( f(x) \) as \( x \) gets closer and closer to 2 from the left.

2. Discuss the behavior of \( f(x) \) as \( x \) gets closer and closer to 2 from the right.
Activity E

Let \( g(x) = \frac{x^2 - 4x - 5}{x + 1} \)

The teacher tells the students that the goal of this activity is to determine the behavior of \( g(x) \) as \( x \) gets closer and closer to 2 from the left and from the right.

Question E1

Tom likes tables.

How would he answer the question?

Demonstrate his strategy.

(Copy the appropriate tables.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td>( x )</td>
<td>( y )</td>
</tr>
</tbody>
</table>

1. As \( x \) gets closer and closer to 2 from the left \( g(x) \) gets closer and closer to ___.

2. As \( x \) gets closer and closer to 2 from the right \( g(x) \) gets closer and closer to ___.

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Question E3
George likes graphs. Use the graphs to answer the following questions.

1. As $x$ gets closer and closer to 2 from the left $g(x)$ gets closer and closer to ______.

2. As $x$ gets closer and closer to 2 from the right $g(x)$ gets closer and closer to ______

Question E2
Alice likes algebraic simplification.
She claims in this case it is appropriate to plug 2 into $g(x)$ in order to determine the behavior of the function as $x$ approaches 2 from the left and from the right.

How would she answer the question? 
Demonstrate her strategy.
Question E3

Carla claims that Alice's strategy of plugging the number into the function sometimes fails. Why does she believe this? Hint: Recall the function from Activity D.

Let \( g(x) = \frac{x^2 - 4x - 5}{x + 1} \). Discuss the behavior of \( g(x) \) as \( x \) gets closer and closer to \(-1\) from the left and from the right.

Question E4

Demonstrate Tom’s strategy. Copy the appropriate tables.

<table>
<thead>
<tr>
<th>X</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>x=-.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>x=-1.1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answers:
- As \( x \) gets closer to \(-1\) from the left, \( f(x) \) gets closer to \__________\.
- As \( x \) gets closer to \(-1\) from the right, \( f(x) \) gets closer to \__________\.
Demonstrate George’s strategy. Press WINDOW and change the numbers to match those below.

\[ y_1 = \frac{(x^2 - 4x - 5)}{(x + 1)} \]

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( y_4 )</th>
<th>( y_5 )</th>
</tr>
</thead>
</table>

\[ \text{WINDOW} \]

\[ \text{Xmin} = -9.4 \]
\[ \text{Xmax} = 9.4 \]
\[ \text{Xscl} = 1 \]
\[ \text{Ymin} = -9.2 \]
\[ \text{Ymax} = 3.2 \]
\[ \text{Yscl} = 1 \]
\[ \text{Xres} = 1 \]

Copy the graph.

How will George answer the questions?

As \( x \) gets closer to \(-1\) from the left, \( f(x) \) gets closer to \_\_\_\_\_.

As \( x \) gets closer to \(-1\) from the right, \( f(x) \) gets closer to \_\_\_\_.

Let \( g(x) = \frac{x^2 - 4x - 5}{x + 1} \). Discuss the behavior of \( g(x) \) as \( x \) gets closer and closer to \(-1\) from the left and from the right.

Carla claims that Alice’s plug-in-the-value strategy fails here. Why does Carla make the claim that one cannot plug \(-1\) into the function?
Question E6

Alice claims that the function $g(x) = \frac{x^2 - 4x - 5}{x + 1}$ is "equal" to another function. What is Alice thinking?

Question E7

Carla believes that Alice's new function could be used with Alice's original "plug in the number" strategy. Demonstrate what Carla is thinking.
The limit of \( f(x) \) as \( x \) approaches 2 from the left is written as \( \lim_{x \to 2^-} f(x) \).

The limit of \( f(x) \) as \( x \) approaches 2 from the right is written as \( \lim_{x \to 2^+} f(x) \).

Informal Definition of a limit

If \( \lim_{x \to c^-} f(x) = L \) and \( \lim_{x \to c^+} f(x) = L \) we say \( \lim_{x \to c} f(x) \) exists and \( \lim_{x \to c} f(x) = L \).

Tom is asked to determine \( \lim_{x \to 2} (x + 3) \).

Show how Tom would solve the problem.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>2.1</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 2} (x + 3) = \]

Why?
George is asked to determine $\lim_{x \to 1} (x^2 - 2x + 4)$.

Explain how George would solve the problem.

\[ \lim_{x \to 1} (x^2 - 2x + 4) = \quad \]

Why?

Alice believes that one could answer Tom’s problem and George’s problem by **plugging-in** numbers. Show how Alice would solve the problems.

1. \[ \lim_{x \to 2} (x + 3) = \quad \]

Why?

2. \[ \lim_{x \to 1} (x^2 - 2x + 4) = \quad \]

Why?

END HOUR 2.
The instructor will present the following rule as a lecture.

**Polynomial Substitution Rule**

Limits of polynomials can be found by substitution. (Thomas 2004, p. 86)

Carla recognizes that Alice’s plugging-in strategy works as long as the function is a

Carla also recalls that a plug-in strategy could be used if two functions were very similar.

The instructor puts the following formal rule on the board

**All-but-One-Point Rule**

Rule: Functions that agree at all but one point (Larson, Hostetler, Edwards 2006, p. 62)

Let \( c \) be a real number and let \( f(x) = g(x) \) for all \( x \neq c \) in an open interval containing \( c \). If the limit of \( g(x) \) as \( x \) approaches \( c \) exists. Then the limit of \( f(x) \) also exists and \( \lim_{x\to c} f(x) = \lim_{x\to c} g(x) \).

Let \( f(x) = \frac{x^2 - 5x + 6}{x - 2} = \frac{(x - 3)(x - 2)}{x - 2} \)

Examine \( \lim_{x\to 2} f(x) \).

This function is VERY similar to the function \( g(x) = x - 3 \).

\( f(x) = g(x) \) everywhere except at the point \( x = 2 \)

Therefore it is fair to use the All-but-One-Point Rule.
Informal version of the All-but-One-Point Rule

If two functions are identical at all but one point, then the limits agree at the missing point.

Since $f(x) = g(x)$ everywhere except at the point $x = 2$ it is fair claim that

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x)$$

As $x$ gets closer and closer to 2, $g(x) = x - 3$ gets closer and closer to $2 - 3 = -1$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2} = \lim_{x \to 2} \frac{(x - 3)(x - 2)}{(x - 2)}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 3)$$

(All-but-One-Point Rule)

$$\lim_{x \to 2} f(x) = 2 - 3$$

(Polynomial Substitution Rule)

$$= -1$$
Reflective Question 3. (Answer after completing the unit.)

Using your own words, carefully describe the following rules. Demonstrate examples where each strategy is appropriate. Relate these concepts to other ideas studied in this unit.

Informal Definition of a Limit

Polynomial Substitution Rule

All-but-One-Point Rule
Exercises: Tom likes tables, George likes graphs, Alice likes substitution rules and Carla likes to look at all strategies. Solve each problem using the substitution rule or the all-but-one-point rule. Check the answers using Tom’s strategy or George’s strategy.

Students will present solutions on the overhead.

1. Let \( f(x) = x^2 + 2x + 1 \). Evaluate \( \lim_{x \to 3} f(x) \).

2. Let \( f(x) = \frac{\sqrt{1+x} - 1}{x} \). Examine whether or not one can find \( \lim_{x \to 0} f(x) \).
3. Let \( f(x) = \frac{x^2 - 4}{x - 2} \). Evaluate \( \lim_{x \to 2} f(x) \).

4. Let \( f(x) = \frac{6 - 3}{2 + x - x} \). Evaluate \( \lim_{x \to 0} f(x) \).

END HOUR 3
Place students in groups to answer the questions. Encourage algebraic strategies.

1. $\lim_{x \to 3} (2x^2 - 3x + 2)$

2. $\lim_{x \to 3} \frac{\sqrt{x+1} - 1}{x - 3}$
3. \( \lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x} \)

4. \( \lim_{x \to 0} \frac{\frac{1}{x + 5} - \frac{1}{5}}{x} \)
Activity F.

Examine \( f(x) = \begin{cases} 
2x + 1 & \text{if } x \leq 1 \\
-x^2 - 3 & \text{if } x > 1 
\end{cases} \)

The instructor claims that this function could be re-written as

If \( x \) is smaller than or equal to 1, then \( f(x) = 2x + 1 \).

If \( x \) is larger than 1, then \( f(x) = x^2 - 3 \).

He says that if \( x \) is a number larger than one that you plug \( x \) into \( x^2 - 3 \) and if \( x \) is smaller than or equal to one you plug \( x \) into \( 2x + 1 \).

Question F1

Alice claims that this is not one function but two.
Tom claims that this is one function written in a strange way.
Carla examines both arguments. What will she decide?

The teacher puts the following examples on the board.

A. Evaluate \( f(2) \).
Since \( 2 > 1 \) we must use the rule \( f(x) = x^2 - 3 \), so \( f(2) = 2^2 - 1 = 4 - 1 = 3 \)

B. Evaluate \( f(-3) \).
Since \( -3 \leq 1 \), we must use the rule \( f(x) = 2x + 1 \), so \( f(-3) = 2(-3) + 1 = -6 + 1 = -5 \).

Tom decides to construct a table. He cannot figure out how to use the table feature of the calculator in this case so he completes the table by hand.
Question F2
Help Tom complete the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>1.001</td>
<td></td>
</tr>
<tr>
<td>1.001</td>
<td></td>
</tr>
</tbody>
</table>

Based on this table answer the following question:

Question F3
As \(x\) gets closer and closer to 1 from the right, \(f(x)\) gets closer and closer to

Question F4
Help Tom complete the following table (by hand):

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>0.999</td>
<td></td>
</tr>
<tr>
<td>0.9999</td>
<td></td>
</tr>
</tbody>
</table>

Based on this table answer the following question:

Question F5
As \(x\) gets closer and closer to 1 from the left, \(f(x)\) gets closer and closer to

Question F6
Carla asks, "Does \(f(x)\) approach the same number as \(x\) gets closer to 1 from the left and from the right?"

Answer: ___________________________
Question F8
Carla says in this case \( \lim_{x \to 1} f(x) \) does not exist.

Explain Carla's reasoning

The students decide to construct a rule. Help the complete the task.

If

\[
\text{______________________________},
\]

then the limit does not exist at \( x = 1 \).
Activity G

The teacher puts the following example on the board

Let \( f(x) = \frac{|x|}{x} \)

The graph follows:

George claims that

as \( x \) approaches 0 from the left, \( f(x) \) approaches -1

and as \( x \) approaches 0 from the right, \( f(x) \) approaches 1.

Question G1

Carla claims that Alice's strategy of plugging the number into the function does not work for this problem. Why does Carla believe this?

Question G2

Alice asks if the limit as \( x \) approaches 0 of \( f(x) = \frac{|x|}{x} \) exists. Carla explains.

Answer: ____________

Why?
The instructor asks the team to examine \( k(x) = \begin{cases} x + 2 & \text{if } x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases} \)

**Question G3**

How would Tom decide what happens to \( k(x) \) as \( x \) gets closer and closer to 2 from the left? Demonstrate how Tom would solve the problem.

\[
\lim_{{x \to 2^-}} k(x) = \underline{\phantom{000000000000}}
\]

**Question G4**

How would Tom decide what happens to \( k(x) \) as \( x \) gets closer and closer to 2 from the right? Demonstrate how Tom would solve the problem.

\[
\lim_{{x \to 2^+}} k(x) = \underline{\phantom{000000000000}}
\]

**Question G5**

Alice claims that the limit as \( x \) approaches 2 of \( k(x) \) does not exist. Explain her reasoning.
Question G6
George is a student in the group who likes using graphs to answer questions. He constructs a graph of \( k(x) = \begin{cases} 
    x + 2 & \text{if } x \leq 2 \\
    x - 3 & \text{if } x > 2
\end{cases} \). Copy his graph.

[Graph of \( k(x) \)]

Question G7
How does George use the graph to determine what happens to \( k(x) \) as \( x \) gets closer and closer to 2 from the left and what happens to \( k(x) \) as \( x \) gets closer and closer to 2 from the right?

a. \( \lim_{x \to 2^-} k(x) = \) 

b. \( \lim_{x \to 2^+} k(x) = \) 

Tom says that \( k(2) = \) 

And he says \( \lim_{x \to 2} k(x) = \) because

---

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Question H6

The instructor asks the team to construct a different function $g(x)$ and its graph such that both of the following statements are true:

As $x$ gets closer and closer to 2 from the left, $g(x)$ gets closer and closer to 1.

\[ \lim_{x \to 2} g(x) = 1 \]

As $x$ gets closer and closer to 2 from the right, $g(x)$ gets closer and closer to 3.

\[ \lim_{x \to 2} g(x) = 3 \]

Construct the graph for the function you created:

```
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
  * * * ■ *
```

Carla asks the group, "Does $\lim_{x \to 2} g(x)$ exist?"

Answer: ____________________________

Why? ____________________________

Carla notices that there is a break in the graph at $x = 2$. She wonders if there is a relationship between the existence of a limit and the connectedness of a graph.
Limits
Hour 5

Activity I
The instructor asks the team to examine the following function: \( g(x) = \frac{-4}{(x - 3)^2} \)

Question 11.

George constructs a graph. His graph follows

George claims as \( x \) gets closer and closer to 3 from the left, \( g(x) \) grows increasingly negative without bound.

George claims as \( x \) gets closer and closer to 3 from the right, \( g(x) \) grows increasingly negative without bound.

Question 12

Alice claims that the limit as \( x \) approaches 3 of \( g(x) \) does not exist. Explain her reasoning.
Question 13
The instructor asks the team to construct a function $q(x)$ and its graph such that both of the following statements are true:

1. As $x$ gets closer and closer to 3 from the left, $q(x)$ gets increasingly positive (without bound).
2. As $x$ gets closer and closer to 3 from the right, $q(x)$ gets increasingly positive (without bound).

Help the team complete the task.

<table>
<thead>
<tr>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q(x) =$ ___________________________</td>
</tr>
</tbody>
</table>

Carla asks the group, "Does $\lim_{x \to 3} q(x)$ exist?"

Answer: ___________________________

Why? ___________________________
Question 14
The instructor asks the team to construct a function \( f(x) \) and its graph such that both of the following statements are true:

1. As \( x \) gets closer and closer to 1 from the left, \( f(x) \) gets increasingly negative (without bound).

2. As \( x \) gets closer and closer to 1 from the right, \( f(x) \) gets increasingly positive numbers (without bound).

Carla asks the group, "Does \( \lim_{x \to 1} g(x) \) exist?"

Answer: _____________________________

Why? _____________________________
Activity J

The instructor asks the team to recall the function \( f(x) = \frac{x^2 - 5x + 6}{x - 2} \)

The students re-examine Tom's tables and the answer the following questions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( X_i )</th>
<th>( Y_i )</th>
<th>( x )</th>
<th>( X_i )</th>
<th>( Y_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.99</td>
<td>-1.01</td>
<td>2.01</td>
<td>-0.99</td>
<td>2.001</td>
<td>-0.999</td>
</tr>
<tr>
<td>1.999</td>
<td>-1.001</td>
<td>2.0001</td>
<td>-0.999</td>
<td>2.00001</td>
<td>-0.9999</td>
</tr>
</tbody>
</table>

Question J1

Use the above tables to answer the questions.

\[
\lim_{x \to 2^-} f(x) =
\]

\[
\lim_{x \to 2^+} f(x) =
\]

\[
\lim_{x \to 2} f(x) =
\]

Question J2

Alice notices that \( f(2) \) does not exist.
Why does Alice believe this?

She believes if \( f(2) \) does not exist, then \( \lim_{x \to 2} f(x) \) should not exist.

What would her team members tell her?
Activity K

Question K1

George is the team member that likes to construct graphs. He wonders, "Can we construct the graph of a function such that

1. \( \lim_{x \to 3} f(x) \) exists

and

2. \( f(3) \) exists

and

3. \( \lim_{x \to 3} f(x) = f(3) ? \)

Help George complete the task,

Without looking at the graph, Tom asks George whether or not the graph is connected at \( x = 3 \).

George's Answer: ______________________________________
Think about It.

Alice wonders if a function \( f \) must be connected at \( x = 3 \) when

\[
limit_{x \to 3} f(x) \text{ exists and } f(3) \text{ exists,}
\]

and \( \lim_{x \to 3} f(x) = f(3) \).

Question K2

George later wonders, "Can we construct the graph of a function such that

1. \( \lim_{x \to 3} f(x) \) exists

and

2. \( f(3) \) exists

but

3. \( \lim_{x \to 3} f(x) \neq f(3) ?" 

Help George complete the task.

Without looking at the graph, Tom asks George whether or not the graph is connected at \( x = 3 \).

George's Answer: ________________________________
Think about It

Alice wonders if a function \( f \) must be disconnected at \( x = 3 \)
if \( \lim_{x \to 3} f(x) \) exists and \( f(3) \) exists, but \( \lim_{x \to 3} f(x) \neq f(3) \)

Question K3

George again wonders, "Can we construct the graph of a function such that as \( x \) gets
closer and closer to 3, \( f(x) \) becomes increasingly negative?" Help George
complete the task.

Think about It

Carla notices that George's first graph (K1) is in one connected piece but his second
and third graphs (K2 and K3) are in disconnected pieces.
The team wonders if the limit has something to do with this fact.
Reflective Question 4 (Answer after completing the unit.)

Explain the relationship between the concept of a limit and the notion of a continuous graph. Use examples and counterexamples in your discussion.
Activity M
Let \( f(x) = \begin{cases} 
  x - 1 & \text{if } x \geq 2 \\
  x^2 - 3 & \text{if } x < 2 
\end{cases} \)

Question M1
Help Tom complete the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.997</td>
<td>0.98801</td>
</tr>
<tr>
<td>1.998</td>
<td>0.992004</td>
</tr>
<tr>
<td>1.999</td>
<td>0.996001</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2.001</td>
<td>1.001</td>
</tr>
<tr>
<td>2.002</td>
<td>1.002</td>
</tr>
<tr>
<td>2.003</td>
<td>1.003</td>
</tr>
</tbody>
</table>

Help the team evaluate each of the following (if possible)

Question M2
\( \lim_{x \to 2} f(x) = \) ____________

Question M3
\( \lim_{x \to 2} f(x) = \) ____________

Question M4
\( f(2) = \) ____________

Question M5
\( \lim_{x \to 2} f(x) = \) ____________
Question M6

George decides to construct a graph for this function. Help him complete the task.

Think about It

Carla notices that the function in this activity is different from the function in the previous activity. She sees that the pieces in George’s graph are connected at \( x = 2 \). Alice asks,

“The pieces in this graph are connected but the pieces in the previous graph were not connected. How is this function different?”
Activity N

The teacher asks the students to examine \( f(x) = \begin{cases} ax + 3 & \text{if } x \leq 1 \\ 2x - 4 & \text{if } x > 1 \end{cases} \)

Question N1
Tom wants to construct a table by hand. Help him complete the task.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.997</td>
<td>( a(0.997) + 3 )</td>
</tr>
<tr>
<td>0.998</td>
<td>( a(0.998) + 3 )</td>
</tr>
<tr>
<td>0.999</td>
<td>( a(0.999) + 3 )</td>
</tr>
<tr>
<td>1</td>
<td>( a(1) + 3 = a + 3 )</td>
</tr>
<tr>
<td>1.001</td>
<td>(-1.998)</td>
</tr>
<tr>
<td>1.002</td>
<td>(-1.996)</td>
</tr>
<tr>
<td>1.003</td>
<td>(-1.994)</td>
</tr>
</tbody>
</table>

Help the team to determine the following

Question N2
\[
\lim_{x \to 1^-} f(x) =
\]

Question N3
\[
\lim_{x \to 1^+} f(x) =
\]

Question N4
Help Carla find a value of \( a \) which enables \( \lim_{x \to 1} f(x) \) to exist.
George decides to change the function from the previous problem. Carla found that \( a = -5 \). So

\[
f(x) = \begin{cases} 
-5x + 3 & \text{if } x \leq 1 \\
2x - 4 & \text{if } x > 1
\end{cases}
\]

by using the value of \( a \) that Carla found.

**Question N6**

Help George graph this new version of \( f(x) \).

\[
f(x) = \begin{cases} 
-5x + 3 & \text{if } x \leq 1 \\
2x - 4 & \text{if } x > 1
\end{cases}
\]

Carla asks the team the following questions. Help the team answer Carla's questions.

**Question N7**

\begin{align*}
a) & \text{ Does } \lim_{x \to 1} f(x) \text{ exist? Answer } \\
b) & \text{ Does } f(1) \text{ exists? Answer } \\
c) & \text{ Does } \lim_{x \to 1} f(x) = f(1) \text{ Answer } \\
d) & \text{ Is the graph "connected" at } x = 1? \text{ Answer }
\end{align*}
Carla again wonders if there is a relationship between connectedness and limits.

Vocabulary

The teacher states that a function is continuous at a point \( x = c \) if the following are true:

1. \( \lim_{x \to c} \) exists
2. \( f(c) \) exists
3. \( \lim_{x \to c} f(x) = f(c) \)

George says that the function \( f(x) = \begin{cases} -5x + 3 & \text{if } x \leq 1 \\ 2x - 4 & \text{if } x > 1 \end{cases} \) from the previous exercise was connected at \( x = 1 \).

Alice says that she can prove it is continuous using the above definition.

Help Alice prove this.
Help the team construct the graph of a function $p(x)$ so that $\lim_{x \to 6} p(x) = -2$ and $p(6) = -2$.

Tom asks whether or not the function is connected at $x = 6$. How would George use the graph answer Tom’s question?

How would Alice use the definition of continuity to answer Tom’s question?
Question P4

Help the team construct the graph of a function \( g(x) \) such that \( \lim_{x \to -3} g(x) = 1 \) and \( \lim_{x \to -3} g(x) = 4 \).

Question P5

Tom asks whether or not the function is continuous at \( x = -3 \).
George likes graphs. How would George answer Tom’s question?

Question P6

Alice likes definitions. How would Alice answer Tom’s question?
Question P7
Help the team construct a graph of a function that
gets increasingly negative as \( x \) approaches 3 from the right
and increasingly positive as \( x \) approaches 3 from the left.

Question P8
Tom asks whether or not the function is continuous at \( x = -3 \).
How would George answer Tom’s question?

Question P9
How would Alice answer Tom’s question?
Activity Q

Let \( f(x) = \begin{cases} x^2 - 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases} \)

Alice wants to determine if the function is continuous at \( x = 0 \). Help her complete her task.
Reflective Question
Carla decides to write a summary of this collection of limit lessons in her notebook. She wants to write definitions in her own words and include relevant examples and counterexamples in her notes. Help Carla complete her task.

What limits mean to me.

END HOUR 5. END LIMIT UNIT
APPENDIX E

CONTROL CURRICULUM
Definition: Left-Hand Limits

\[ \lim_{x \to c^-} f(x) \] means if \( f(x) \) is defined on an interval \((a, c)\), where \( a < c \) and \( f(x) \) gets closer and closer to \( M \) as \( x \) gets closer and closer to \( c \) from within that interval, then \( f \) has left-hand limit \( M \) at \( c \) and we write \( \lim_{x \to c^-} f(x) = M \).

Example 1: \( f(x) = x^2 + 2x + 1 \). Evaluate \( \lim_{x \to 3} f(x) \)

The graph of the function follows.

We must ask the question “as \( x \) gets closer and closer to three from the left, what does \( f(x) \) get closer and closer to?”

I shall examine the “Left-Half” of the function, that is the graph of \( f(x) \) when \( x < 3 \).
Question?

What happens to the values of \( f(x) \) as \( x \) gets closer and closer to 3 from the left?

If this trend were to continue, what will the \( y \)-values get closer and closer to?

I will zoom in on the graph to get a better idea of how the function is behaving.

If this trend continues, what should happen at \( x = 3 \)?

The graph indicates that the \( y \)-values should get closer and closer to 16. Therefore

It seems reasonable that as \( x \) gets closer and closer to 3 from the left, \( f(x) \) or \( y \) gets closer and closer to 16.

So

\[
\lim_{x \to 3^-} f(x) = 16
\]
Example 2
Let \( g(x) = \frac{x^2 - 4}{x + 2} \). Find \( \lim_{x \to -2} g(x) \).

The following is a graph of \( g(x) \).

We shall examine the "Left-Half" of the graph, that is the graph of \( g(x) \) for \( x < -2 \).

Zooming in we see

Graphical analysis indicates that that \( \lim_{x \to -2} g(x) = -4 \).
Recall that $g(x) = \frac{x^2 - 4}{x + 2}$ so $g(-2)$ does not exist, but $\lim_{x \to -2} g(x)$ does exist and it equals $-4$. 
Example 3

Examine \( f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 1 \\ x^2 - 3 & \text{if } x > 1 \end{cases} \)

Find \( \lim_{x \to 1} f(x) \).

This means

If \( x \) is smaller than or equal to 1, then \( f(x) = 2x + 1 \).

If \( x \) is larger than 1, then \( f(x) = x^2 - 3 \)

Evaluate \( f(-3) \).

Since \(-3 \leq 1\), we must use the rule \( f(x) = 2x + 1 \), so \( f(-3) = 2(-3) + 1 = -6 + 1 = -5 \).

Construct the half of the graph for \( x < 1 \). In this case we must use the rule \( f(x) = 2x + 1 \)

What happens to the values of \( f(x) \) as \( x \) gets closer and closer to 1 from the left?

Zooming in we get

As \( x \) gets closer and closer to 1 from the left it seems that \( f(x) \) gets closer and closer to 3.
Therefore \( \lim_{x \to c} f(x) = 3 \)

---

**Definition: Right-hand limits**

Define (informally) limits from above (or from the right). \( \lim_{x \to c^+} f(x) \) means if \( f(x) \) is defined on an interval \((c, b)\), where \( c < b \) and \( x \) gets closer and closer to \( L \) as \( x \) gets closer and closer to \( c \) from within that interval, then \( f \) has right-hand limit \( L \) at \( c \) and we write \( \lim_{x \to c^+} f(x) = L \)

---

**Example 4:** \( f(x) = x^2 + 2x + 1 \). Evaluate \( \lim_{x \to 3^+} f(x) \).

The graph of the function follows.

![Graph](image)

We must ask the question “as \( x \) gets closer and closer to three from the left, what does \( f(x) \) get closer and closer to?”

I shall examine the “Right-Half” of the function, that is the graph of \( f(x) \) when \( x > 3 \).
What happens to the values of $f(x)$ as $x$ gets closer and closer to 3 from the right?

As $x$ gets closer and closer to 3 from the right, it appears that $f(x)$ gets closer and closer to 16.

Therefore $\lim_{x \to 3} f(x) = 3$
Example 5

Let \( g(x) = \frac{x^2 - 4}{x + 2} \). Find \( \lim_{x \to -2} g(x) \).

The following is a graph of \( g(x) \).

We shall examine the "Right-Half" of the graph, that is the graph of \( g(x) \) for \( x > -2 \).

Zooming in we see

Graphical analysis indicates that \( \lim_{x \to -2} g(x) = -4 \).
Recall that $g(x) = \frac{x^2 - 4}{x + 2}$ so $g(-2)$ does not exist, but $\lim_{x \to -2} g(x)$ does exist and it equals $-4$.

Example 6

Examine $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 1 \\ x^2 - 3 & \text{if } x > 1 \end{cases}$ Find $\lim_{x \to 1} f(x)$.

This means

If $x$ is smaller than or equal to 1, then $f(x) = 2x + 1$.
If $x$ is larger than 1, then $f(x) = x^2 - 3$.

Find $\lim_{x \to 1} f(x)$.

Construct the half of the graph for $x > 1$. In this case we must use the rule $f(x) = x^2 - 3$.

What happens to the values of $f(x)$ as $x$ gets closer and closer to 1 from the left?

Zooming in we see the following:
Therefore it appears that $\lim_{x \to c} f(x) = -2$

Informal Definition of a limit

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = L$ we say $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$.

Since $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(x) = 16$,
we can conclude that $\lim_{x \to 3^-} f(x)$ exists and $\lim_{x \to 3^-} f(x) = 16$.

Example 7

Let $f(x) = x^2 + 2x + 1$. Evaluate $\lim_{x \to 3} f(x)$.

First find $\lim_{x \to 3} f(x)$

As $x$ gets closer and closer to 3 from the left, $f(x)$ gets closer and closer to 16.
Therefore $\lim_{x \to 3^-} f(x) = 16$. 
First find $\lim_{x \to 3} f(x)$

As $x$ gets closer and closer to 3 from the right, $f(x)$ gets closer and closer to 16. So $\lim_{x \to 3} f(x) = 16$.

Since $\lim_{x \to 3} f(x) = \lim_{x \to 3} f(x) = 16$, we conclude $\lim_{x \to 3} f(x) = 16$.

Example 8

Let $g(x) = \frac{x^2 - 4}{x + 2}$. Find $\lim_{x \to -2} g(x)$.

Zooming in we see

Graphical analysis indicates that $\lim_{x \to -2} g(x) = -4$.
Zooming in we see

\[
\begin{align*}
&\text{Graphical analysis indicates that } \lim_{x \to -2} g(x) = -4. \\
\text{Since } \lim_{x \to -2} g(x) &= \lim_{x \to -2} g(x) = -4,
\end{align*}
\]

We conclude that
\[
\lim_{x \to -2} g(x) \text{ exists and } \lim_{x \to -2} g(x) = -4
\]
Example 9: Let \( f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 1 \\ x^2 - 3 & \text{if } x > 1 \end{cases} \)

Determine \( \lim_{x \to 1} f(x) \)

The graph follows:

First find \( \lim_{x \to 1^-} f(x) \) by examining the "Left-Half" of the graph.

Therefore it appears that \( \lim_{x \to 1^-} f(x) = 3 \)

Next find \( \lim_{x \to 1^+} f(x) \) by examining the "Right-Half" of the graph.

Therefore it appears that \( \lim_{x \to 1^+} f(x) = -2 \)

Determine \( \lim_{x \to 1} f(x) \)

Since \( \lim_{x \to 1^-} f(x) = 3 \) and \( \lim_{x \to 1^+} f(x) = -2 \) we conclude that \( \lim_{x \to 1} f(x) \) does not exist

END HOUR ONE
Properties of Limits

The following rules hold if \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \) (\( L \) and \( M \) are real numbers).

1. Identity Rule: \( \lim_{x \to c} x = c \)

2. Constant Rule: \( \lim_{x \to c} k = k \) (any real number \( k \))

3. Sum Rule: \( \lim_{x \to c} (f(x) + g(x)) = L + M \)

4. Difference Rule: \( \lim_{x \to c} (f(x) - g(x)) = L - M \)

5. Product Rule: \( \lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M \)

6. Constant Multiple Rule: \( \lim_{x \to c} (k f(x)) = k L \) (any real number \( k \))

7. Quotient Rule: \( \lim_{x \to c} \left( \frac{f(x)}{g(x)} \right) = \frac{L}{M} \) \( (M \neq 0) \)

8. Power Rule: If \( m \) and \( n \) are integers, then \( \lim_{x \to c} (f(x))^{m/n} = L^{m/n} \) provided \( L^{m/n} \) is a real number.
Example 10

Evaluate \( \lim_{x \to \infty} \frac{x^3 + 4x^2 - 3}{x^2 + 5} \) using the properties of limits.

\[
\lim_{x \to \infty} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{\lim_{x \to \infty} (x^3 + 4x^2 - 3)}{\lim_{x \to \infty} (x^2 + 5)} \quad \text{(Rule 7)}
\]

\[
= \frac{\left( \lim_{x \to \infty} x^3 + \lim_{x \to \infty} 4x^2 - \lim_{x \to \infty} 3 \right)}{\left( \lim_{x \to \infty} x^2 + \lim_{x \to \infty} 5 \right)} \quad \text{(Rules 3 and 4)}
\]

\[
= \frac{\left( \lim_{x \to \infty} x^3 + 4 \lim_{x \to \infty} x^2 - \lim_{x \to \infty} 3 \right)}{\left( \lim_{x \to \infty} x^2 + \lim_{x \to \infty} 5 \right)} \quad \text{(Rule 6)}
\]

\[
= \frac{\left( \lim_{x \to \infty} x \cdot \lim_{x \to \infty} x \cdot \lim_{x \to \infty} x + 4 \cdot \lim_{x \to \infty} x \cdot \lim_{x \to \infty} x - \lim_{x \to \infty} 3 \right)}{\left( \lim_{x \to \infty} x \cdot \lim_{x \to \infty} x + \lim_{x \to \infty} 5 \right)} \quad \text{(Rule 5)}
\]

\[
= \frac{\left( c \cdot c \cdot c + 4c \cdot c - \lim_{x \to \infty} 3 \right)}{\left( c \cdot c + \lim_{x \to \infty} 5 \right)} \quad \text{(Rule 1)}
\]

\[
= \frac{\left( c^3 + 4c^2 - 3 \right)}{\left( c^2 + 5 \right)} \quad \text{(Rule 2)}
\]
Example 11: Evaluate $\lim_{x \to 2} \sqrt{4x^2 - 3}$

\[
\lim_{x \to 2} \sqrt{4x^2 - 3} = \left[ \lim_{x \to 2} (4x^2 - 3) \right]^{1/2}
\]

(Rule 8)

\[
= \left[ \lim_{x \to 2} 4x^2 - \lim_{x \to 2} 3 \right]^{1/2}
\]

(Rule 4)

\[
= \left[ 4 \lim_{x \to 2} x^2 - \lim_{x \to 2} 3 \right]^{1/2}
\]

(Rule 6)

\[
= \left[ 4 \left( \lim_{x \to 2} x \right)^2 \left( \lim_{x \to 2} x \right) - \lim_{x \to 2} 3 \right]^{1/2}
\]

(Rule 5)

\[
= \left[ 4(-2)(-2) - \lim_{x \to 2} 3 \right]^{1/2}
\]

(Rule 1)

\[
= [16 - 3]^{1/2}
\]

(Rule 2)

\[
= \sqrt{13}
\]

Polynomial Substitution Rule

Limits of polynomials can be found by substitution.

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_0$ then

$$
\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + a_{n-2} c^{n-2} + \cdots + a_0
$$

If $f(x) = P(x)$ on an open interval containing $c$ then

$$
\lim_{x \to c} f(x) = \lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + a_{n-2} c^{n-2} + \cdots + a_0
$$

Example 12

Let $f(x) = x^2 + 2x + 1$. Evaluate $\lim_{x \to 3} f(x)$.

Since $f(x) = x^2 + 2x + 1$ is a polynomial we can apply the polynomial substitution rule.

$$
\lim_{x \to 3} f(x) = f(3) = 3^2 + 2 \cdot 3 + 1 = 16.
$$

Example 13
Let \( f(x) = \begin{cases} 
2x+1 & \text{if } x \leq 1 \\
x^2-3 & \text{if } x > 1
\end{cases} \)

Examine \( \lim_{x \to 1} f(x) \).

This is a piecewise function that is composed of two polynomials. It is NOT a polynomial. It is NOT a polynomial on any interval containing the point \( x = 1 \). It is not fair to apply the polynomial substitution rule.

We know \( f(1) = 2(1) + 1 = 3 \)

We also have shown \( \lim_{x \to 1} f(x) \) does not exist.

Therefore \( \lim_{x \to 1} f(x) \neq f(1) \).

Example 14
Let \( f(x) = \begin{cases} 
2x+1 & \text{if } x \leq 1 \\
x^2-3 & \text{if } x > 1
\end{cases} \)

Examine \( \lim_{x \to -4} f(x) \).

This is a piecewise function that is NOT a polynomial, however, near the point \( x = -4 \), \( f(x) \) behaves like the polynomial \( p(x) = 2x + 1 \). Since \( f(x) = p(x) = 2x + 1 \) on an interval containing \( x = -4 \), we conclude

\[
\lim_{x \to -4} f(x) = \lim_{x \to -4} p(x) = p(-4) = 2(-4) + 1 = -8 + 1 = -7
\]
Rational Substitution Rule
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{{x \to c}} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$

Example 15: Let $k(x) = \frac{6}{x+1}$. Find $\lim_{{x \to 2}} k(x)$

$$k(x) = \frac{P(x)}{Q(x)} = \frac{6}{x+1}$$
Since $Q(2) \neq 0$ the substitution rule applies.

$$\lim_{{x \to 2}} k(x) = \frac{P(2)}{Q(2)} = \frac{6}{2+1} = \frac{6}{3} = 2$$

Example 16:
Again let $k(x) = \frac{6}{x+1}$. Evaluate $\lim_{{x \to 1}} k(x)$.

$$k(x) = \frac{P(x)}{Q(x)} = \frac{6}{x+1}$$
but $Q(-1) = 0$, therefore we cannot apply the rational function substitution rule to determine $\lim_{{x \to -1}} k(x)$.

We see why one cannot use the plug in rule for a limit by examining the graph.

To find $\lim_{{x \to -1}} k(x)$ we look at the "Left-Half" of the graph, that is the graph of $k(x)$ for $x < -1$.
As \( x \) gets closer and closer to \(-1\) from the left, we see an asymptote on the graph at \( x = -1 \). The values of \( f(x) \) seem to becoming larger and larger negative numbers.

The table demonstrates what happens as \( x \) gets closer and closer to \(-1\) from the left.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( k(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.1</td>
<td>-60</td>
</tr>
<tr>
<td>-1.01</td>
<td>-600</td>
</tr>
<tr>
<td>-1.001</td>
<td>-6000</td>
</tr>
<tr>
<td>-1.0001</td>
<td>-60000</td>
</tr>
</tbody>
</table>

The values of \( k(x) \) become larger and larger negative numbers.

There is no real number \( M \) that \( f(x) \) gets closer and closer to so we conclude that that \( \lim_{x \to -1} k(x) \) does not exist.

Since \( \lim_{x \to -1} k(x) \) does not exist, we can conclude that \( \lim_{x \to -1} k(x) \) does not exist.

Notice that we can also show that \( \lim_{x \to -1} k(x) \) does not exist.
As $x$ gets closer and closer to $-1$ from the right, we see an asymptote on the graph at $x = -1$. The values of $f(x)$ seem to becoming larger and larger positive numbers.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>60</td>
</tr>
<tr>
<td>-0.99</td>
<td>600</td>
</tr>
<tr>
<td>-0.999</td>
<td>6000</td>
</tr>
<tr>
<td>-0.9999</td>
<td>60000</td>
</tr>
</tbody>
</table>

There is no real number $M$ that $f(x)$ gets closer and closer to, so we conclude that $\lim_{x \to -1} k(x)$ does not exist.

If $\lim_{x \to -1} k(x)$ does not exist we can conclude $\lim_{x \to -1} k(x)$ does not exist.

If $\lim_{x \to -1} k(x)$ does not exist we can conclude $\lim_{x \to -1} k(x)$ does not exist.

We do not need both to be true in order to conclude that $\lim_{x \to -1} k(x)$ does not exist.
Example 17

\[ f(x) = \frac{x^2 - 4x - 5}{x + 1} \]

Find \( \lim_{x \to 2} f(x) \)

The substitution rule for rational functions applies here because

\[ f(x) = \frac{P(x)}{Q(x)} = \frac{x^2 - 4x - 5}{x + 1} \quad \text{and} \quad Q(2) = 2 + 1 = 3 \neq 0. \]

Therefore \( \lim_{x \to 2} f(x) = \frac{P(2)}{Q(2)} = \frac{2^2 - 4 \cdot 2 - 5}{2 + 1} = \frac{4 - 8 - 5}{3} = \frac{-9}{3} = -3 \)

If time remains, answer homework questions.

END HOUR 2.
Limits
Hour 3

Answer homework questions at the start of the hour.

All-but-One-Point Rule

Rule: Functions that agree at all but one point (Larson, Hostetler, Edwards 1994, p. 76)
Let \( c \) be a real number and let \( f(x) = g(x) \) for all \( x \neq c \) in an open interval containing \( c \). If the limit of \( g(x) \) as \( x \) approaches \( c \) exists. Then the limit of \( f(x) \) also exists and \( \lim_{{x \to c}} f(x) = \lim_{{x \to c}} g(x) \).

Example 18
Let \( f(x) = \frac{(x^2 - 5x + 6)}{(x - 2)} = \frac{(x - 3)(x - 2)}{(x - 2)} \)

Examine \( \lim_{{x \to 2}} f(x) \).

Notice \( f(x) = \frac{(x^2 - 5x + 6)}{(x - 2)} = \frac{P(x)}{Q(x)} \) so \( f(x) \) is a rational function, but \( Q(2) = 0 \) so the Rational substitution does not apply if you want to find \( \lim_{{x \to 2}} f(x) \).

But we can use algebraic simplification to re-examine \( f(x) \).

\[
 f(x) = \frac{(x^2 - 5x + 6)}{(x - 2)} = \frac{(x - 3)(x - 2)}{(x - 2)}
\]

This function is VERY similar to the function \( g(x) = x - 3 \).
\( f(x) = g(x) \) everywhere except at the point \( x = 2 \)

Therefore it is fair to use the All-but-One-Point Rule

\[
 \lim_{{x \to 2}} f(x) = \text{but } g(x) \text{ is a polynomial so that } \lim_{{x \to 2}} g(x) = g(2).
\]

\[
 \lim_{{x \to 2}} f(x) = \lim_{{x \to 2}} g(x) = g(2) = g(2) = 2 - 3 = -1
\]

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Example 19
Let \( g(x) = \frac{x^2 - 4x - 5}{x + 1} \). Evaluate \( \lim_{x \to -1} g(x) \)

\[
g(x) = \frac{x^2 - 4x - 5}{x + 1} = \frac{(x-5)(x+1)}{(x+1)}
\]

So \( g(x) \) is the same function as \( h(x) = x - 5 \) except at the value \( x = -1 \).

Therefore the All but One Point rule applies.

\[
\lim_{x \to -1} g(x) = \lim_{x \to -1} \frac{x^2 - 4x - 5}{x + 1} = \lim_{x \to -1} \frac{(x-5)(x+1)}{(x+1)} = \lim_{x \to -1} (x-5) = -1 - 5 = -6
\]

Example 20
Let \( f(x) = \frac{\sqrt{1+x} - 1}{x} \). Examine whether or not we can find \( \lim_{x \to 0} f(x) \).

Notice \( f(0) \) does not exist.

Let \( g(x) = \sqrt{1+x} - 1 \), then \( g(x) = \frac{\sqrt{1+x} + 1}{x} = \frac{1 + x - 1}{x(\sqrt{1+x} + 1)} = \frac{x}{x(\sqrt{1+x} + 1)} = \frac{1}{\sqrt{1+x} + 1} \)

So, if \( \lim_{x \to 0} g(x) \) exists, then \( \lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) \) because the functions agree at all but one point.

So \( \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1+x} + 1} \)

Notice that the denominator is no longer zero when \( x = 0 \). Even though this function is not a quotient of two polynomials, it is fair to extend this rule to cases like this when the denominator is not zero.

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{1+1} = \frac{1}{2}
\]

Answer homework questions

END HOUR 3
Hour 4
Definition of Continuity
A function is **continuous** at a point \( x = c \) if the following are true

i) \( \lim_{x \to c} f(x) \) exists

ii) \( f(c) \) exists

iii) \( \lim_{x \to c} f(x) = f(c) \)

Example 21
Examine \( f(x) = 3x - 2 \). Is \( f(x) \) continuous at \( x = 0 \)?

\[
\lim_{x \to 0} f(x) = f(0) = -2. \quad \text{(Polynomial substitution Rule)}
\]

\( f(0) = 3 \cdot 0 - 2 = -2 \)

Since \( \lim_{x \to 0} f(x) = f(0) \) we conclude that \( f \) is continuous at \( x = 0 \).

By examining the graph we see that there is no break in the graph at \( x = 0 \)

![Graph of f(x) = 3x - 2](image)

This reinforces the notion that \( f(x) \) is continuous at \( x = 0 \).
Example 22

Let \( g(x) = \frac{x^2 - 4}{x + 2} \). Is \( g(x) \) continuous at \( x = -2 \)?

The following is a graph of \( g(x) \).

\[
\lim_{x \to -2^-} g(x) = -4 \quad \text{and} \quad \lim_{x \to -2^+} g(x) = -4. \quad \text{So} \quad \lim_{x \to -2} g(x) = -4.
\]

But \( g(2) \) does not exist, so \( g(x) \) is not continuous at \( x = -2 \).

Notice if we define a new function

\[
h(x) = \begin{cases} 
   \frac{x^2 - 4}{x + 2}, & \text{if } x \neq -2 \\
   -4, & \text{if } x = -2
\end{cases}
\]

\[
\lim_{x \to -2} h(x) = -4 \quad \text{and} \quad h(-2) = -4, \quad \text{so this function is continuous at } x = -2.
\]
Example 23

Construct the graph of a function \( g(x) \) such that as \( x \) gets closer and closer to 4 from below, \( g(x) \) gets closer and closer to 2 and as \( x \) gets closer and closer to 4 from above, \( g(x) \) gets closer and closer to 2. Also let \( g(4) = 2 \)

Based on the definition, we know that \( \lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x) = \lim_{x \to 4} g(x) = 2 \)

Since \( \lim_{x \to 4} g(x) = g(4) = 2 \) we know that \( g(x) \) must be continuous at \( x = 4 \).

An example of a function of this type would be \( g(x) = \left(\frac{1}{2}\right)x \).

The graph indicates that this function is continuous at \( x = 4 \).
Example 24

Let \( f(x) = \frac{|x|}{x} \). Determine whether or not \( f(x) \) is continuous at \( x = 0 \).

The graph follows:

Questions:
Use the graph to determine
\( \text{a) } \lim_{x \to 0^-} f(x) \)
\( \text{b) } \lim_{x \to 0^+} f(x) \)
\( \text{c) } \lim_{x \to 0} f(x) \)

Answers
From the graph we can see that \( \lim_{x \to 0^-} f(x) = 1 \) and \( \lim_{x \to 0^+} f(x) = -1 \), therefore \( \lim_{x \to 0} f(x) \) does not exist. Also \( f(0) \) does not exist.

Since \( \lim_{x \to 0} f(x) \) does not exist, we know that the function \( f(x) = \frac{|x|}{x} \) is not continuous at \( x = 0 \).
Example 25

\[ f(x) = \begin{cases} 
  x + 2 & \text{if } x \leq 2 \\
  x - 3 & \text{if } x > 2 
\end{cases} \]

Determine whether or not \( f(x) \) is continuous at \( x = 2 \).

a) Construct a graph of the function.

b) Determine \( \lim_{{x \to 2^+}} k(x) \)

It appears from the graph that \( \lim_{{x \to 2^+}} k(x) = -1 \)

c) Determine \( \lim_{{x \to 2^-}} k(x) \)

It appears from the graph that \( \lim_{{x \to 2^-}} k(x) = 3 \)

d) Determine \( \lim_{{x \to 2^+}} k(x) \)

Since \( \lim_{{x \to 2^-}} k(x) \neq \lim_{{x \to 2^+}} k(x) \) we conclude that \( \lim_{{x \to 2}} k(x) \) does not exist.

e) \( k(2) = 2 + 2 = 4 \)

Conclusion: Since \( \lim_{{x \to 2}} k(x) \) does not exist, the function \( k(x) \) is not continuous at \( x = 2 \).

If time remains, answer homework questions.

END HOUR 4
Limits
Hour 5
Recall the definition of continuity

A function is **continuous** at a point \( x = c \) if the following are true

i) \( \lim_{x \to c} f(x) \) exists

ii) \( f(c) \) exists

iii) \( \lim_{x \to c} f(x) = f(c) \)

Example 26
Let \( g(x) = \frac{-4}{(x-3)^2} \)

i.) If possible find \( \lim_{x \to 3} f(x) \).

ii.) Determine whether or not this function is continuous at \( x = 3 \).

a) Construct a graph

![Graph of \( g(x) \)]

b) Determine \( \lim_{x \to 3} g(x) \)

From the graph it appears that as \( x \) gets closer and closer to 3 from the right that \( g(x) \) becomes increasingly large (negative) numbers. Therefore \( \lim_{x \to 3} g(x) \) does not exist.

c) Determine \( \lim_{x \to 3} g(x) \)

From the graph it appears that as \( x \) gets closer and close to 3 from the left that \( g(x) \) becomes increasingly large (negative) numbers. Therefore \( \lim_{x \to 3} g(x) \) does not exist.

d) Determine \( \lim_{x \to 3} g(x) \)

\( \lim_{x \to 3} g(x) \) does not exist. In fact if only ONE of b) or c) did not exist, the conclusion would be the same.
e) Determine $g(3)$.

We see that $g(3)$ does not exist.

Conclusion: Since $\lim_{x \to 3} g(x)$ does not exist, we conclude that $g(x)$ is not continuous at $x = 3$.

We also could have first looked at $g(3)$. Since $g(3)$ does not exist, we can conclude that $g(x)$ is not continuous at $x = 3$.

Example 27

Let $f(x) = \begin{cases} 3x & \text{if } x < 1 \\ 5 - 2x & \text{if } x > 1 \end{cases}$

If possible, find $\lim_{x \to 1} f(x)$ and then determine whether or not $f(x)$ is continuous at $x = 1$.

First find $\lim_{x \to 1} f(x)$. Look at the "Left-Half" of the graph.

We recognize as $x < 1$ we must use the rule $f(x) = 3x$, so as $x$ gets closer and closer to 1 from the left we see that $f(x)$ gets closer and closer to 3. So $\lim_{x \to 1} f(x) = 3$.
We recognize as $x > 1$ we must use the rule $f(x) = 5 - 2x$, so as $x$ gets closer and closer to 1 from the left we see that $f(x)$ gets closer and closer to 3.

So $\lim_{x \to 1^-} f(x) = 3$.

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1} f(x) = 3$, we conclude $\lim_{x \to 1} f(x) = 3$.

But $f(x) = \begin{cases} 3x & \text{if } x < 1 \\ 5 - 2x & \text{if } x > 1 \end{cases}$

1 is not in the domain of the function, so $f(1)$ does not exist.

Therefore $f$ is not continuous at $x = 1$. 

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Example 28

Let \( f(x) = \begin{cases} 
ax + 3 & \text{if } x \leq 1 \\
2x - 4 & \text{if } x > 1 
\end{cases} \)

Find a value for \( a \) so that the function \( f(x) \) is continuous at \( x = 1 \).

A. First find \( \lim_{x \to 1^-} f(x) \).

If \( x \leq 1 \) we must use the rule \( f(x) = ax + 3 \), so as \( x \) gets closer and closer to \( 1 \) from the left, \( f(x) \) gets closer and closer to \( a(1) + 3 = a + 3 \).

Therefore \( \lim_{x \to 1^-} f(x) = a + 3 \)

B. Find \( \lim_{x \to 1^+} f(x) \).

If \( x > 1 \) we must use the rule \( f(x) = 2x - 4 \), so as \( x \) gets closer and closer to \( 1 \) from the right, \( f(x) \) gets closer and closer to \( 2(1) - 4 = 2 - 4 = -2 \).

Therefore \( \lim_{x \to 1^+} f(x) = -2 \)

C. Find \( \lim_{x \to 1} f(x) \).

We want the limit to exist so we need \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \)

So \( a + 3 = -2 \Rightarrow a = -5 \)

If \( a = -5 \), then \( \lim_{x \to 1} f(x) = -2 \)

D. Find \( f(1) \). If we let \( a = -5 \) we can re-write \( f(x) \) as follows:

\( f(x) = \begin{cases} 
-5x + 3 & \text{if } x \leq 1 \\
2x - 4 & \text{if } x > 1 
\end{cases} \)

\( f(1) = (-5) \cdot 1 + 3 = -5 + 3 = -2 \)

Conclusion

Since \( \lim_{x \to 1} f(x) = f(1) = -2 \), we know that \( f(x) \) is continuous at \( x = 1 \).

If time remains,

1. Review concepts from earlier in the week.

2. Answer homework questions.

END HOUR 5. END LIMIT UNIT