

Value Distribution of L-Functions with Rational Moving Targets

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ABSTRACT

We prove some value-distribution results for a class of L -functions with rational moving targets. The class contains Selberg class, as well as the Riemann-zeta function.

Keywords: Value Distribution; Moving Target; L -Function; Selberg Class

1. Introduction

We define the class \mathcal{M} to be the collection of functions $L(s) = \sum_{n=1}^{\infty} a(n)/n^s$, satisfying Ramanujan hypothesis, Analytic continuation and Functional equation. We also denote the degree of a function $L \in \mathcal{M}$ by d_L which is a non-negative real number. We refer the reader to Chapter six of [1] for a complete definitions. Obviously, the class \mathcal{M} contains the Selberg class. Also every function in the class \mathcal{M} is an L -function and the Riemann-zeta function is in the class. In this paper, we prove a value-distribution theorem for the class \mathcal{M} with rational moving targets. The theorem generalizes the value-distribution results in Chapter seven of [1] from fixed targets to moving targets.

Theorem. Assume that $L \in \mathcal{M}$ and R is a rational function with $\lim_{s \rightarrow \infty} R(s) \neq 1$. Let the roots of the equation $L(s) - R(s) = 0$ be denoted by $\rho_R = \beta_R + i\gamma_R$. Then

$$(I) \text{ For any } b > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\},$$

$$\sum_{\substack{\beta_R > b \\ T < \gamma_R \leq 2T}} (\beta_R - b) = O(T), \text{ as } T \rightarrow \infty.$$

(II) For sufficiently large negative b ,

$$2\pi \sum_{T < \gamma_R \leq 2T} (\beta_R - b) = (-b)d_L T \log \frac{4T}{e} + O(\log T),$$

as $T \rightarrow \infty$.

Proof of (I). It is known that if $L \in \mathcal{M}$, then

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = 1 + O(k_0^{-\sigma}), \text{ as } \sigma \rightarrow \infty;$$

where k_0 is the index of the first non-zero term of the sequence of $\{a(n)\}_{n=2}^{\infty}$, $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Since $\lim_{\sigma \rightarrow \infty} L(s) - R(s) \neq 0$, there exists $\sigma_0 > 0$ such that $L(s) - R(s) \neq 0$ for $\text{Re } s = \sigma > \sigma_0$. It follows that $\beta_R < \sigma_0$ for all real part of zeros of the function $L(s) - R(s)$. We set $R(z) = P(z)/Q(z)$ where the degrees of P, Q are p, q , respectively; and define

$$\tilde{\ell}(s) = (s) - R(s).$$

Thus, there is $r_1 > 1$ such that $\tilde{\ell}$ is analytic in the region $|s| > r_1$ since L is a meromorphic function in \mathbb{C} with the only pole at $s = 1$. We apply Littlewood's argument principle [3] to $\tilde{\ell}$ in the rectangle $\mathcal{R} = \{\sigma + it : b \leq \sigma \leq c, T \leq t \leq 2T\}$ where c, T are parameters satisfying $c > \max\{\sigma_0 + 1, b\}, T > r_1$. Thus,

$$\int_{\partial \mathcal{R}} \log \tilde{\ell}(s) ds = -2\pi i \int_b^c \nu(\sigma, \mathcal{R}) d\sigma$$

where the given logarithm is defined as in Littlewood's argument principle [3]. To prove our result, however, we first decompose our auxiliary function by

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$$\tilde{\ell}(s) = \begin{cases} P(s) \left(\frac{L(s)}{P(s)} - \frac{1}{Q(s)} \right) := P(s)\ell_1(s) & \text{for } p \leq q \\ R(s) \left(\frac{L(s)}{R(s)} - 1 \right) := R(s)\ell_2(s) & \text{for } p > q \end{cases} \quad (1)$$

Without loss of generality, we may assume that $p, q \geq 1$ whenever $p \leq q$ since we can always write

$$\int_{\partial\mathcal{R}} \log \tilde{\ell}(s) ds = \begin{cases} \int_{\partial\mathcal{R}} \log \ell_1(s) + \log P(s) ds + O(T) & \text{for } p \leq q \\ \int_{\partial\mathcal{R}} \log \ell_2(s) + \log R(s) ds + O(T) & \text{for } p > q \end{cases}$$

where the $O(T)$ terms are the integrals of the maximum contribution from writing $\tilde{\ell}(s)$ as a sum of logarithms. By our choice of T , both $\log P$ and $\log R$ are analytic in \mathcal{R} . Hence, Cauchy's Theorem gives

$$\int_{\partial\mathcal{R}} \log \tilde{\ell}(s) ds = \begin{cases} \int_{\partial\mathcal{R}} \log \ell_1(s) ds + O(T) & \text{for } p \leq q \\ \int_{\partial\mathcal{R}} \log \ell_2(s) ds + O(T) & \text{for } p > q \end{cases} \quad (2)$$

To connect this integral with Littlewood's argument principle [3], we note that the definition of c guaran-

$$\begin{aligned} & 2\pi i \sum_{\substack{\beta_R > b \\ T < \gamma_R \leq 2T}} (\beta_R - b) \\ &= i \operatorname{Im} \left[\int_b^c \log |\ell_k(\sigma + iT)| + i \arg \ell_k(\sigma + iT) d\sigma + i \int_T^{2T} \log |\ell_k(c + it)| + i \arg \ell_k(c + it) dt \right. \\ & \quad \left. - \int_b^c \log |\ell_k(\sigma + 2iT)| + i \arg \ell_k(\sigma + 2iT) d\sigma - i \int_T^{2T} \log |\ell_k(b + it)| + i \arg \ell_k(b + it) dt \right] + O(T) \\ &= -i \left[\int_T^{2T} \log |\ell_k(b + it)| dt - \int_T^{2T} \log |\ell_k(c + it)| dt - \int_b^c \arg \ell_k(\sigma + iT) d\sigma + \int_b^c \arg \ell_k(\sigma + 2iT) d\sigma \right] + O(T) \\ &:= \sum_{j=1}^4 I_{j,k} + O(T), \end{aligned} \quad (3)$$

for instance.

We now estimate $I_{1,k}$. For T large enough, we have for $t \geq T, k = 1$ (since $p, q \geq 1$),

$$\begin{aligned} \log |\ell_1(b + it)| &= \log \left| \frac{L(b + it)}{P(b + it)} - \frac{1}{Q(b + it)} \right| \leq \log \left(\left| \frac{L(b + it)}{P(b + it)} \right| + \left| \frac{1}{Q(b + it)} \right| \right) \\ &\leq \log (|L(b + it)| + 1) = \log^+ (|L(b + it)| + 1) \leq \log^+ |L(b + it)| + \log 2. \end{aligned}$$

Then for T large enough, $t \geq T, k = 2$, we find in a similar fashion that

$$\begin{aligned} \log |\ell_2(b + it)| &= \log \left| \frac{L(b + it)}{R(b + it)} - 1 \right| \\ &\leq \log^+ (|L(b + it)|) + \log 2. \end{aligned}$$

Since we have the same estimate for $k = 1, 2$, we find that

$R(s) = (P(s)s^N)/(Q(s)s^N)$ for $s \neq 0$ due to our choice of the parameters which define the rectangle \mathcal{R} . However, the modification will guarantee in the case of $k = 1$ that P, Q exhibit polynomial growth, which is necessary for our proof. In the case of $p > q$, R already exhibits polynomial growth, and no such adjustment is necessary. We now integrate the logarithm of $\tilde{\ell}$ to get

tees that

$$\begin{aligned} -2\pi i \int_b^c \nu(\sigma, \mathcal{R}) d\sigma &= -2\pi i \sum_{\substack{\beta_R > b \\ T < \gamma_R \leq 2T}} \int_b^{\beta_R} d\sigma \\ &= -2\pi i \sum_{\substack{\beta_R > b \\ T < \gamma_R \leq 2T}} (\beta_R - b). \end{aligned} \quad (4)$$

In light of (2) and because the quantity given in (3) is imaginary-valued, we get for $k = 1, 2$

$$\begin{aligned} I_{1,k}(T, b) &= I_{1,k} \leq \int_T^{2T} \log^+ (|L(b + it)|) dt + O(T) \\ &= \frac{T}{2} \int_T^{2T} \frac{\log^+ (|L(b + it)|)^2}{T} dt + O(T) \\ &\leq \frac{T}{2} \log^+ \left(\frac{1}{T} \int_T^{2T} |L(b + it)|^2 dt \right) + O(T) \end{aligned}$$

where the final bound follows from Jensen's inequality.

It is known [2] that for $b > \max\left\{\frac{1}{2}, 1 - \frac{1}{d_L}\right\}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_T^{2T} |L(b+it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^{2\sigma}} = O(1).$$

Hence, $I_{1,k}(T, b) \leq O(T)$ uniformly in

$$b > \max\left\{\frac{1}{2}, 1 - \frac{1}{d_L}\right\}.$$

We next move to estimate $I_{2,k}$. For sufficiently large positive real number c , we have

$$\left| \frac{L(c+it)}{P(c+it)} \right| \leq 1 \text{ and } \left| \frac{L(c+it)}{R(c+it)} \right| \leq 1, \tag{5}$$

so

$$\log |\ell_1(c+it)| \leq \log \left| 1 - \frac{L(c+it)}{P(c+it)} \right|$$

since $q \geq 1$. Furthermore,

$$\log |\ell_2(c+it)| = \left| 1 - \frac{L(c+it)}{R(c+it)} \right|.$$

Since we may take c large enough so that $|\ell_k(c+it)| \leq 1$, we may write $\log \ell_k(c+it)$ using a Taylor series expansion in the rectangle \mathcal{R} . For $k=1$, we have after taking real parts that

$$\begin{aligned} \log |\ell_1(c+it)| &\leq \operatorname{Re} \left(- \sum_{k=1}^{\infty} \frac{1}{k [P(c+it)]^k} \left(\sum_{n=1}^{\infty} \frac{a(n)}{n^{c+it}} \right)^k \right) \\ &= - \operatorname{Re} \left(\sum_{k=1}^{\infty} \frac{1}{k [P(c+it)]^k} \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} \frac{a(n_1) \dots a(n_k)}{(n_1 \dots n_k)^{c+it}} \right). \end{aligned}$$

We now observe that for sufficiently large T and some constant M we have

$$\int_T^{2T} \left| \frac{t}{[P(c+it)]^k (n_1 \dots n_k)^{it}} \right| \leq \frac{T}{|P(c+iT)|^k} \leq MT^{1-k} \leq 1,$$

for $k \in N$ and

$$\limsup_{k \rightarrow \infty} \sqrt[k]{\frac{1}{k} \left(\sum_{n=1}^{\infty} \frac{1}{n^{c-\varepsilon}} \right)^k} = \sum_{n=1}^{\infty} \frac{1}{n^{c-\varepsilon}} < 1$$

for sufficiently large c . In light of these bounds and the definition of \mathcal{M} , we have (6)

where the last equality holds because c could be sufficiently large. Replacing P by R in the above computations, we see analogously that $|I_{2,2}| = O(1)$.

Finally, we estimate $I_{3,k}$ and $I_{4,k}$. We show the computation for $I_{3,k}$ explicitly and note that the bound for $I_{4,k}$ follows analogously. We first suppose that $\ell_k(\sigma+iT)$ has exactly N zeros for $b \leq \sigma \leq c$. Then, there are at most $N+1$ subintervals, counting for multiplicities, in which $\operatorname{Re}(\ell_k(\sigma+iT))$ is of constant sign. Thus,

$$\left| \arg(\ell_k(\sigma+iT)) \right| \leq (N+1)\pi. \tag{7}$$

It remains to estimate N . To this end, we define

$$g_k(z) = \frac{1}{2} \left(\ell_k(z+iT) + \overline{\ell_k(\bar{z}+iT)} \right).$$

Then

$$g_k(\sigma) = \frac{1}{2} \left(\ell_k(\sigma+iT) + \overline{\ell_k(\sigma+iT)} \right) = \operatorname{Re} \ell_k(\sigma+iT),$$

so that if $\ell_k(\sigma+iT) = 0$ for $\sigma \in [b, c]$, then $g(\sigma) = 0$.

Now let $R_2 = c - b$ and $R' > \max\{r_1, R_2\}$, and choose T large enough so that $T > 2R'$. Then $|z+iT| > R' > R_1$ for $|z-c| < R'$, showing that no zeros or poles of $\ell_k(z+iT)$ are located in $|z-c| < R'$. Thus, both $\ell_k(z+iT)$ and $g_k(z)$ are analytic in $|z-c| < R'$. Letting $\hat{n}_{c,k}(r)$ denote the number of zeros of $g_k(z)$ in $|z-c| \leq r$, we have

$$\begin{aligned} \int_0^{2R'} \frac{\hat{n}_{c,k}(r)}{r} dr &\geq \int_{R'}^{2R'} \frac{\hat{n}_{c,k}(r)}{r} dr \\ &\geq \hat{n}_{c,k}(R') \int_{R'}^{2R'} \frac{dr}{r} = \hat{n}_{c,k}(R') \log 2. \end{aligned}$$

By Jensen's formula

$$\int_0^{2R'} \frac{\hat{n}_{c,k}(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g_k(c+2R'e^{i\theta})| d\theta - \log |g_k(c)|,$$

and so

$$\hat{n}_{c,k}(R') \leq \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g_k(c+2R'e^{i\theta})| d\theta - \frac{\log |g_k(c)|}{\log 2}. \tag{8}$$

$$\begin{aligned} |I_{2,1}| &= \left| - \operatorname{Re} \left(\sum_{k=1}^{\infty} \frac{1}{k} \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} \frac{a(n_1) \dots a(n_k)}{(n_1 \dots n_k)^c} \int_T^{2T} \frac{dt}{[P(c+it)]^k (n_1 \dots n_k)^{it}} \right) \right| \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} \left| \frac{a(n_1) \dots a(n_k)}{(n_1 \dots n_k)^c} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{n=1}^{\infty} \frac{1}{n^{c-\varepsilon}} \right)^k = O(1), \end{aligned} \tag{6}$$

By (5), $\log|g_k(c)|$ is bounded. Further, it is clear from a property of L functions that we have

$$|L(s)| \leq A|t|^B, \text{ as } t \rightarrow \infty, \text{ as } t \rightarrow \infty;$$

for some positive absolute numbers A, B in any vertical strip of bounded width. The same estimate must hold for $g_k(z)$ as well. Thus, the integral in (8) is $O(\log T)$, implying that $\hat{n}_{c,k}(R) = O(\log T)$. Since the interval $[b, c] \subseteq D(c, R_2) \subseteq D(c, R')$, it follows that

$$N \leq \hat{n}_{c,k}(R') = O(\log T).$$

With this bound, we integrate (7) to deduce that

$$|I_{3,k}| \leq \int_b^c |\arg \ell_k(\sigma + it)| d\sigma \leq \int_b^c (N + 1)\pi d\sigma = O(\log T).$$

As previously noted, we may bound $I_{4,k}$ in the same

$$\begin{aligned} 2\pi i \sum_{T < \gamma_R \leq 2T} (\beta_R - b) &= -i \left[\int_T^{2T} \log|\tilde{\ell}(b + it)| dt - \int_T^{2T} \log|\ell_k(c + it)| dt \right] \\ &= \int_b^c \arg \ell_k(\sigma + iT) d\sigma + \int_b^c \arg \ell_k(\sigma + 2iT) d\sigma + O(T) \\ &:= I_1 + \sum_{j=2}^4 I_{j,k} + O(T) \end{aligned}$$

for $k=1,2$ where ℓ_k is defined as in (1). In the equation above, we note that we have chosen to compute I_1 separately. Indeed, this is the only estimate that we will need. For the integrals $I_{j,k}$, $j=2,3,4$ and $k=1,2$, the bounds given as in the proof of the first part of the theorem still hold. First, integral $I_{2,k}$ is

$$\begin{aligned} |L(s) - R(s)| &= |\Lambda_L(s) \overline{L(1-\bar{s})} - R(s)| = |\Lambda_L(s)| |L(1-\bar{s})| \left| 1 - \frac{R(s)}{\Lambda_L(s)L(1-\bar{s})} \right| \\ &= |\Lambda_L(s)| |L(1-\bar{s})| \left| 1 - \frac{R(s)}{L(s)} \right|. \end{aligned}$$

Taking logarithms, we get

$$\begin{aligned} \log|L(s) - R(s)| &= \log|\Lambda_L(s)| + \log|L(1-\bar{s})| + \log \left| 1 - \frac{R(s)}{L(s)} \right|. \end{aligned} \tag{9}$$

Since, for $t > 1$, we have, uniformly in σ ,

$$\begin{aligned} \log|\Lambda_L(s)| &= \log \left| (\lambda Q^2 t^{d_L})^{\frac{1}{2}-\sigma-it} \exp \left(it + \frac{i\pi(\mu-d_L)}{4} \right) \left(1 + O\left(\frac{1}{t}\right) \right) \right| \leq \left(\frac{1}{2} - \sigma \right) \log|\lambda Q^2 t^{d_L}| \\ &\quad - \log \left| (\lambda Q^2 t^{d_L})^it \right| + \log \left| 1 + O\left(\frac{1}{t}\right) \right| = \left(\frac{1}{2} - \sigma \right) (d_L \log t + \log(\lambda Q^2)) + O\left(\frac{1}{t}\right). \end{aligned}$$

way. Thus, we attain the desired bounds for $j=1, \dots, 4$ and $k=1, 2$. Consequently, the first part of the theorem is proved by using (4).

Proof of (II). As in the proof of the first part of the theorem, we conclude that there exists a real number σ_0 for which the real parts β_R of all R -values satisfy $\beta_R < \sigma_0$; and also, there exist $B, T' > 0$ for each rational function R such that no zeros of $L(s) - R(s) = 0$ lie in the quarter-plane $\sigma < -B, t > T'$.

As before, we define the rectangle $\mathcal{R} = \{s = \sigma + it : b \leq \sigma \leq c, T \leq t \leq 2T\}$ where b, c, T are parameters satisfying $b < -B - 1, c > \max\{\sigma_0 + 1, b\}, T > \max\{T', T' + 1\}$.

Proceeding as in the proof of the first part of the theorem, we see that

unchanged. On the other hand, the integrals $I_{3,k}, I_{4,k}$ have changed by our choice of b , but, as we have done as before, we still have the desired bound since the only requirement is that we consider L in a vertical strip of fixed width, which we have in this case.

We now bound I_1 . Since $b < -B$, we have by the functional equation in the definition of L function,

$$\begin{aligned} \log|\Lambda_L(s)| &= (\lambda Q^2 t^{d_L})^{\frac{1}{2}-\sigma-it} \exp \left(it d_L + \frac{i\pi(\mu-d_L)}{4} \right) \left(1 + O\left(\frac{1}{t}\right) \right), \end{aligned}$$

where μ, λ are two constants. It follows, for $s = \sigma + it$ as $t \rightarrow \infty$, that

We now consider the last term in (9). Since,

$$\limsup_{t \rightarrow \pm\infty} \frac{\log |L(b+it)|}{\log |t|} = \left(\frac{1}{2} - b\right) d_L,$$

and noting $b < 0$, we have for any $\delta > 0$ and $t \geq T$

$$|L(b+it)| \geq |t|^{\left(\frac{1}{2}-b\right)d_L-\delta}$$

for sufficiently large T . Then we see the quotient

$$\left| \frac{R(b+it)}{L(b+it)} \right| \leq \left| \frac{R(b+it)}{|t|^{\left(\frac{1}{2}-b\right)d_L-\delta}} \right| = O\left(\frac{1}{t}\right)$$

when $-b$ is large enough so that

$$\deg R < \left(\frac{1}{2} - b\right) d_L - \delta + 1.$$

Therefore, we find that

$$\log \left| 1 - \frac{R(s)}{L(s)} \right| = O\left(\frac{1}{t}\right).$$

Integrating in light of these estimates, we see

$$\begin{aligned} & \int_T^{2T} \log |L(b+it) - R(b+it)| dt \\ &= \left(\frac{1}{2} - b\right) \int_T^{2T} (d_L \log t + \log(\lambda Q^2)) dt \\ & \quad + \int_T^{2T} \log |L(1-b-it)| dt + O(\log T). \end{aligned}$$

The first integral is $d_L T \log \frac{4T}{e} + T \log(\lambda Q^2)$, and the second integral is $O(1)$ for sufficiently large and negative b by the method used to derive (6). Hence,

$$I_1 = \left(\frac{1}{2} - b\right) \left(d_L T \log \frac{4T}{e} + T \log(\lambda Q^2) \right) + O(\log T).$$

With the estimates for the $I_{j,k}$'s, we have proved the second part of the theorem.

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