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Value Distribution of L-Functions with Rational Moving Targets

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ABSTRACT

We prove some value-distribution results for a class of *L*-functions with rational moving targets. The class contains Selberg class, as well as the Riemann-zeta function.

Keywords: Value Distribution; Moving Target; *L*-Function; Selberg Class

1. Introduction

We define the class $\mathcal{M}$ to be the collection of functions $L(s)=\sum_{n=1}^{\infty}a(n)/n^s$, satisfying Ramanujan hypothesis, Analytic continuation and Functional equation. We also denote the degree of a function $L\in\mathcal{M}$ by d_L which is a non-negative real number. We refer the reader to Chapter six of [1] for a complete definitions. Obviously, the class $\mathcal{M}$ contains the Selberg class. Also every function in the class $\mathcal{M}$ is an L-function and the Riemann-zeta function is in the class. In this paper, we prove a value-distribution theorem for the class $\mathcal{M}$ with rational moving targets. The theorem generalizes the value-distribution results in Chapter seven of [1] from fixed targets to moving targets.

Theorem. Assume that $L\in\mathcal{M}$ and R is a rational function with $\lim_{s\to\infty}R(s)\neq 1$. Let the roots of the equation $L(s)-R(s)=0$ be denoted by $\rho_R=\beta_R+i\gamma_R$. Then

(I) For any $b>\max\left\{\frac{1}{2},1-\frac{1}{d_L}\right\}$,

$$\sum_{\substack{\beta_R>b\\ T<\gamma_R\le 2T}}(\beta_R-b)=O(T),\quad \text{as } T\to\infty.$$

(II) For sufficiently large negative b,

$$2\pi\sum_{T<\gamma_R\le 2T}(\beta_R-b)=(-b)d_L T\log\frac{4T}{e}+O(\log T),$$

as $T\to\infty$.

Proof of (I). It is known that if $L\in\mathcal{M}$, then

$$L(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=1+O\left(k_0^{-\sigma}\right),\quad \text{as } \sigma\to\infty;$$

where k_0 is the index of the first non-zero term of the sequence of $\{a(n)\}_{n=2}^{\infty}$, $s=\sigma+it$ with $\sigma,t\in\mathbb{R}$. Since $\lim_{\sigma\to\infty}L(s)-R(s)\neq 0$, there exists $\sigma_0>0$ such that $L(s)-R(s)\neq 0$ for $\operatorname{Re}s=\sigma>\sigma_0$. It follows that $\beta_R<\sigma_0$ for all real part of zeros of the function $L(s)-R(s)$. We set $R(z)=P(z)/Q(z)$ where the degrees of P,Q are p,q, respectively; and define

$$\tilde{\ell}(s)=(s)-R(s).$$

Thus, there is $r_1>1$ such that $\tilde{\ell}$ is analytic in the region $|s|>r_1$ since L is a meromorphic function in $\mathbb{C}$ with the only pole at $s=1$. We apply Littlewood's argument principle [3] to $\tilde{\ell}$ in the rectangle $\mathcal{R}=\{\sigma+it: b\le\sigma\le c, T\le t\le 2T\}$ where c,T are parameters satisfying $c>\max\{\sigma_0+1,b\}, T>r_1$. Thus,

$$\int_{\partial\mathcal{R}}\log\tilde{\ell}(s)\,\mathrm{d}s=-2\pi i\int_b^c\nu(\sigma,\mathcal{R})\,\mathrm{d}\sigma$$

where the given logarithm is defined as in Littlewood's argument principle [3]. To prove our result, however, we first decompose our auxiliary function by

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$$\tilde{\ell}(s)=\begin{cases}P(s)\left(\dfrac{L(s)}{P(s)}-\dfrac{1}{Q(s)}\right):=P(s)\ell_1(s) & \text{for } p\le q\\[2ex] R(s)\left(\dfrac{L(s)}{R(s)}-1\right):=R(s)\ell_2(s) & \text{for } p>q\end{cases} \tag{1}$$

Without loss of generality, we may assume that $p,q\ge 1$ whenever $p\le q$ since we can always write $R(s)=\left(P(s)s^N\right)/\left(Q(s)s^N\right)$ for $s\ne 0$ due to our choice of the parameters which define the rectangle $\mathcal{R}$. However, the modification will guarantee in the case of $k=1$ that P,Q exhibit polynomial growth, which is necessary for our proof. In the case of $p>q$, R already exhibits polynomial growth, and no such adjustment is necessary. We now integrate the logarithm of $\tilde{\ell}$ to get

$$\int_{\partial\mathcal{R}}\log\tilde{\ell}(s)\,\mathrm{d}s=\begin{cases}\int_{\partial\mathcal{R}}\log\ell_1(s)+\log P(s)\,\mathrm{d}s+O(T) & \text{for } p\le q\\ \int_{\partial\mathcal{R}}\log\ell_2(s)+\log R(s)\,\mathrm{d}s+O(T) & \text{for } p>q\end{cases}$$

where the $O(T)$ terms are the integrals of the maximum contribution from writing $\tilde{\ell}(s)$ as a sum of logarithms. By our choice of T, both $\log P$ and $\log R$ are analytic in $\mathcal{R}$. Hence, Cauchy's Theorem gives

$$\int_{\partial\mathcal{R}}\log\tilde{\ell}(s)\,\mathrm{d}s=\begin{cases}\int_{\partial\mathcal{R}}\log\ell_1(s)\,\mathrm{d}s+O(T) & \text{for } p\le q\\ \int_{\partial\mathcal{R}}\log\ell_2(s)\,\mathrm{d}s+O(T) & \text{for } p>q\end{cases} \tag{2}$$

To connect this integral with Littlewood's argument principle [3], we note that the definition of c guarantees that

$$\begin{aligned}-2\pi i\int_b^c \nu(\sigma,\mathcal{R})\,\mathrm{d}\sigma &= -2\pi i\sum_{\substack{\beta_R>b\\ T<\gamma_R\le 2T}}\int_b^{\beta_R}\mathrm{d}\sigma\\ &= -2\pi i\sum_{\substack{\beta_R>b\\ T<\gamma_R\le 2T}}\left(\beta_R-b\right).\end{aligned} \tag{3}$$

In light of (2) and because the quantity given in (3) is imaginary-valued, we get for $k=1,2$

$$\begin{aligned}&2\pi i\sum_{\substack{\beta_R>b\\ T<\gamma_R\le 2T}}\left(\beta_R-b\right)\\ &= i\,\mathrm{Im}\Big[\int_b^c\log\left|\ell_k(\sigma+iT)\right|+i\arg\ell_k(\sigma+iT)\,\mathrm{d}\sigma+i\int_T^{2T}\log\left|\ell_k(c+it)\right|+i\arg\ell_k(c+it)\,\mathrm{d}t\\ &\quad-\int_b^c\log\left|\ell_k(\sigma+2iT)\right|+i\arg\ell_k(\sigma+2iT)\,\mathrm{d}\sigma-i\int_T^{2T}\log\left|\ell_k(b+it)\right|+i\arg\ell_k(b+it)\,\mathrm{d}t\Big]+O(T)\\ &= -i\Big[\int_T^{2T}\log\left|\ell_k(b+it)\right|\mathrm{d}t-\int_T^{2T}\log\left|\ell_k(c+it)\right|\mathrm{d}t-\int_b^c\arg\ell_k(\sigma+iT)\,\mathrm{d}\sigma+\int_b^c\arg\ell_k(\sigma+2iT)\,\mathrm{d}\sigma\Big]+O(T)\\ &:= \sum_{j=1}^4 I_{j,k}+O(T),\end{aligned} \tag{4}$$

for instance.

We now estimate $I_{1,k}$. For T large enough, we have for $t\ge T, k=1$ (since $p,q\ge 1$),

$$\begin{aligned}\log\left|\ell_1(b+it)\right| &= \log\left|\frac{L(b+it)}{P(b+it)}-\frac{1}{Q(b+it)}\right|\le\log\left(\left|\frac{L(b+it)}{P(b+it)}\right|+\frac{1}{\left|Q(b+it)\right|}\right)\\ &\le\log\left(\left|L(b+it)\right|+1\right)=\log^+\left(\left|L(b+it)\right|+1\right)\le\log^+\left|L(b+it)\right|+\log 2.\end{aligned}$$

Then for T large enough, $t\ge T, k=2$, we find in a similar fashion that

$$\begin{aligned}\log\left|\ell_2(b+it)\right| &= \log\left|\frac{L(b+it)}{R(b+it)}-1\right|\\ &\le\log^+\left|(b+it)\right|+\log 2.\end{aligned}$$

Since we have the same estimate for $k=1,2$, we find that

$$\begin{aligned}I_{1,k}(T,b)=I_{1,k} &\le \int_T^{2T}\log^+\left|(b+it)\right|\mathrm{d}t+O(T)\\ &= \frac{T}{2}\int_T^{2T}\frac{\log^+\left|(b+it)\right|^2}{T}\mathrm{d}t+O(T)\\ &\le \frac{T}{2}\log^+\left(\frac{1}{T}\int_T^{2T}\left|L(b+it)\right|^2\mathrm{d}t\right)+O(T)\end{aligned}$$

where the final bound follows from Jensen's inequality.

It is known [2] that for $b > \max\left\{\frac{1}{2}, 1-\frac{1}{d_L}\right\}$,

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}\left|L(b+it)\right|^2 \mathrm{d}t = \sum_{n=1}^{\infty}\frac{|a(n)|^2}{n^{2\sigma}} = O(1).$$

Hence, $I_{1,k}(T,b) \le O(T)$ uniformly in $b > \max\left\{\frac{1}{2}, 1-\frac{1}{d_L}\right\}$.

We next move to estimate $I_{2,k}$. For sufficiently large positive real number c, we have

$$\left|\frac{L(c+it)}{P(c+it)}\right| \le 1 \text{ and } \left|\frac{L(c+it)}{R(c+it)}\right| \le 1, \tag{5}$$

so

$$\log\left|\ell_1(c+it)\right| \le \log\left|1-\frac{L(c+it)}{P(c+it)}\right|$$

since $q \ge 1$. Furthermore,

$$\log\left|\ell_2(c+it)\right| = \left|1-\frac{L(c+it)}{R(c+it)}\right|.$$

Since we may take c large enough so that $\left|\ell_k(c+it)\right| \le 1$, we may write $\log \ell_k(c+it)$ using a Taylor series expansion in the rectangle $\mathcal{R}$. For $k=1$, we have after taking real parts that

$$\log\left|\ell_1(c+it)\right| \le \mathrm{Re}\left(-\sum_{k=1}^{\infty}\frac{1}{k\left[P(c+it)\right]^k}\left(\sum_{n=1}^{\infty}\frac{a(n)}{n^{c+it}}\right)^k\right)$$

$$= -\mathrm{Re}\left(\sum_{k=1}^{\infty}\frac{1}{k\left[P(c+it)\right]^k}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{a(n_1)\cdots a(n_k)}{(n_1\cdots n_k)^{c+it}}\right).$$

We now observe that for sufficiently large T and some constant M we have

$$\int_T^{2T}\left|\frac{t}{\left[P(c+it)\right]^k (n_1\cdots n_k)^{it}}\right| \le \frac{T}{\left|P(c+iT)\right|^k} \le MT^{1-k} \le 1,$$

for $k \in N$ and

$$\limsup_{k\to\infty}\sqrt[k]{\frac{1}{k}\left(\sum_{n=1}^{\infty}\frac{1}{n^{c-\varepsilon}}\right)^k} = \sum_{n=1}^{\infty}\frac{1}{n^{c-\varepsilon}} < 1$$

for sufficiently large c. In light of these bounds and the definition of $\mathcal{M}$, we have (6)

$$\left|I_{2,1}\right| = \left|-\mathrm{Re}\left(\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{a(n_1)\cdots a(n_k)}{(n_1\cdots n_k)^c}\int_T^{2T}\frac{\mathrm{d}t}{\left[P(c+it)\right]^k (n_1\cdots n_k)^{it}}\right)\right|$$
$$\le \sum_{k=1}^{\infty}\frac{1}{k}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\left|\frac{a(n_1)\cdots a(n_k)}{(n_1\cdots n_k)^c}\right| \le \sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{n=1}^{\infty}\frac{1}{n^{c-\varepsilon}}\right)^k = O(1), \tag{6}$$

where the last equality holds because c could be sufficiently large. Replacing P by R in the above computations, we see analogously that $\left|I_{2,2}\right| = O(1)$.

Finally, we estimate $I_{3,k}$ and $I_{4,k}$. We show the computation for $I_{3,k}$ explicitly and note that the bound for $I_{4,k}$ follows analogously. We first suppose that $\ell_k(\sigma+iT)$ has exactly N zeros for $b \le \sigma \le c$. Then, there are at most $N+1$ subintervals, counting for multiplicities, in which $\mathrm{Re}\left(\ell_k(\sigma+iT)\right)$ is of constant sign. Thus,

$$\left|\arg\left(\ell_k(\sigma+iT)\right)\right| \le (N+1)\pi. \tag{7}$$

It remains to estimate N. To this end, we define

$$g_k(z) = \frac{1}{2}\left(\ell_k(z+iT) + \overline{\ell_k(\overline{z}+iT)}\right).$$

Then

$$g_k(\sigma) = \frac{1}{2}\left(\ell_k(\sigma+iT) + \overline{\ell_k(\sigma+iT)}\right) = \mathrm{Re}\,\ell_k(\sigma+iT),$$

so that if $\ell_k(\sigma+iT) = 0$ for $\sigma \in [b,c]$, then $g(\sigma) = 0$.

Now let $R_2 = c-b$ and $R' > \max\{r_1, R_2\}$, and choose T large enough so that $T > 2R'$. Then $|z+iT| > R' > R_1$ for $|z-c| < R'$, showing that no zeros or poles of $\ell_k(z+iT)$ are located in $|z-c| < R'$. Thus, both $\ell_k(z+iT)$ and $g_k(z)$ are analytic in $|z-c| < R'$. Letting $\hat{n}_{c,k}(r)$ denote the number of zeros of $g_k(z)$ in $|z-c| \le r$, we have

$$\int_0^{2R'}\frac{\hat{n}_{c,k}(r)}{r}\mathrm{d}r \ge \int_{R'}^{2R'}\frac{\hat{n}_{c,k}(r)}{r}\mathrm{d}r$$
$$\ge \hat{n}_{c,k}(R')\int_{R'}^{2R'}\frac{\mathrm{d}r}{r} = \hat{n}_{c,k}(R')\log 2.$$

By Jensen's formula

$$\int_0^{2R'}\frac{\hat{n}_{c,k}(r)}{r}\mathrm{d}r = \frac{1}{2\pi}\int_0^{2\pi}\log\left|g_k\left(c+2R'\mathrm{e}^{i\theta}\right)\right|\mathrm{d}\theta - \log\left|g_k(c)\right|,$$

and so

$$\hat{n}_{c,k}(R') \le \frac{1}{2\pi\log 2}\int_0^{2\pi}\log\left|g_k\left(c+2R'\mathrm{e}^{i\theta}\right)\right|\mathrm{d}\theta - \frac{\log\left|g_k(c)\right|}{\log 2}. \tag{8}$$

By (5), $\log|g_k(c)|$ is bounded. Further, it is clear from a property of L functions that we have

$$|L(s)| \le A|t|^B, \text{as } t \to \infty, \text{as } t \to \infty;$$

for some positive absolute numbers A, B in any vertical strip of bounded width. The same estimate must hold for $g_k(z)$ as well. Thus, the integral in (8) is $O(\log T)$, implying that $\hat{n}_{c,k}(R) = O(\log T)$. Since the interval $[b,c] \subseteq D(c,R_2) \subseteq D(c,R')$, it follows that

$$N \le \hat{n}_{c,k}(R') = O(\log T).$$

With this bound, we integrate (7) to deduce that

$$|I_{3,k}| \le \int_b^c |\arg \ell_k(\sigma+it)| \mathrm{d}\sigma \le \int_b^c (N+1)\pi \mathrm{d}\sigma = O(\log T).$$

As previously noted, we may bound $I_{4,k}$ in the same way. Thus, we attain the desired bounds for $j = 1,\cdots,4$ and $k = 1,2$. Consequently, the first part of the theorem is proved by using (4).

Proof of (II). As in the proof of the first part of the theorem, we conclude that there exists a real number σ_0 for which the real parts β_R of all R-values satisfy $\beta_R < \sigma_0$; and also, there exist $B, T' > 0$ for each rational function R such that no zeros of $L(s) - R(s) = 0$ lie in the quarter-plane $\sigma < -B, t > T'$. As before, we define the rectangle $\mathcal{R} = \{s = \sigma + it : b \le \sigma \le c, T \le t \le 2T\}$ where b, c, T are parameters satisfying $b < -B-1, c > \max\{\sigma_0 + 1, b\}, T > \max\{r_1, T'+1\}$.

Proceeding as in the proof of the first part of the theorem, we see that

$$\begin{aligned}2\pi i \sum_{T<\gamma_R \le 2T} (\beta_R - b) &= -i\Big[\int_T^{2T} \log|\tilde{\ell}(b+it)| \mathrm{d}t - \int_T^{2T} \log|\ell_k(c+it)| \mathrm{d}t \\ &= \int_b^c \arg \ell_k(\sigma+iT)\mathrm{d}\sigma + \int_b^c \arg \ell_k(\sigma+2iT)\mathrm{d}\sigma\Big] + O(T) \\ &:= I_1 + \sum_{j=2}^{4} I_{j,k} + O(T)\end{aligned}$$

for $k = 1,2$ where ℓ_k is defined as in (1). In the equation above, we note that we have chosen to compute I_1 separately. Indeed, this is the only estimate that we will need. For the integrals $I_{j,k}$, $j = 2,3,4$ and $k = 1,2$, the bounds given as in the proof of the first part of the theorem still hold. First, integral $I_{2,k}$ is unchanged. On the other hand, the integrals $I_{3,k}, I_{4,k}$ have changed by our choice of b, but, as we have done as before, we still have the desired bound since the only requirement is that we consider L in a vertical strip of fixed width, which we have in this case.

We now bound I_1. Since $b < -B$, we have by the functional equation in the definition of L function,

$$\begin{aligned}|L(s) - R(s)| &= \left|\Lambda_L(s)\overline{L(1-\bar{s})} - R(s)\right| = |\Lambda_L(s)||L(1-\bar{s})|\left|1 - \frac{R(s)}{\Lambda_L(s)L(1-\bar{s})}\right| \\ &= |\Lambda_L(s)||L(1-\bar{s})|\left|1 - \frac{R(s)}{L(s)}\right|.\end{aligned}$$

Taking logarithms, we get

$$\begin{aligned}&\log|L(s) - R(s)| \\ &= \log|\Lambda_L(s)| + \log\left|\overline{L(1-\bar{s})}\right| + \log\left|1 - \frac{R(s)}{L(s)}\right|.\end{aligned}$$

Since, for $t > 1$, we have, uniformly in σ,

$$\begin{aligned}&\log|\Lambda_L(s)| \\ &= \left(\lambda Q^2 t^{d_L}\right)^{\frac{1}{2}-\sigma-it} \exp\left(itd_L + \frac{i\pi(\mu - d_L)}{4}\right)\left(1 + O\left(\frac{1}{t}\right)\right),\end{aligned} \tag{9}$$

where μ, λ are two constants. It follows, for $s = \sigma + it$ as $t \to \infty$, that

$$\begin{aligned}\log|\Lambda_L(s)| &= \log\left|\left(\lambda Q^2 t^{d_L}\right)^{\frac{1}{2}-\sigma-it} \exp\left(it + \frac{i\pi(\mu - d_L)}{4}\right)\left(1 + O\left(\frac{1}{t}\right)\right)\right| \le \left(\frac{1}{2} - \sigma\right)\log\left|\lambda Q^2 t^{d_L}\right| \\ &\quad - \log\left|\left(\lambda Q^2 t^{d_L}\right)^{it}\right| + \log\left|1 + O\left(\frac{1}{t}\right)\right| = \left(\frac{1}{2} - \sigma\right)\left(d_L \log t + \log\left(\lambda Q^2\right)\right) + O\left(\frac{1}{t}\right).\end{aligned}$$

We now consider the last term in (9). Since,

$$\limsup_{t\to\pm\infty}\frac{\log\left|L(b+it)\right|}{\log|t|}=\left(\frac{1}{2}-b\right)d_L,$$

and noting $b<0$, we have for any $\delta>0$ and $t\geq T$

$$\left|L(b+it)\right|\geq|t|^{\left(\frac{1}{2}-b\right)d_L-\delta}$$

for sufficiently large T. Then we see the quotient

$$\left|\frac{R(b+it)}{L(b+it)}\right|\leq\left|\frac{R(b+it)}{t^{\left(\frac{1}{2}-b\right)d_L-\delta}}\right|=O\left(\frac{1}{t}\right)$$

when $-b$ is large enough so that

$$\deg R<\left(\frac{1}{2}-b\right)d_L-\delta+1.$$

Therefore, we find that

$$\log\left|1-\frac{R(s)}{L(s)}\right|=O\left(\frac{1}{t}\right).$$

Integrating in light of these estimates, we see

$$\begin{aligned}&\int_T^{2T}\log\left|L(b+it)-R(b+it)\right|\mathrm{d}t\\&=\left(\frac{1}{2}-b\right)\int_T^{2T}\left(d_L\log t+\log\left(\lambda Q^2\right)\right)\mathrm{d}t\\&\quad+\int_T^{2T}\log\left|L(1-b-it)\right|\mathrm{d}t+O(\log T).\end{aligned}$$

The first integral is $d_L T\log\frac{4T}{e}+T\log\left(\lambda Q^2\right)$, and the second integral is $O(1)$ for sufficiently large and negative b by the method used to derive (6). Hence,

$$I_1=\left(\frac{1}{2}-b\right)\left(d_L T\log\frac{4T}{e}+T\log\left(\lambda Q^2\right)\right)+O(\log T).$$

With the estimates for the $I_{j,k}$'s, we have proved the second part of the theorem.

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