Application of Time Series Models (ARIMA, GARCH, and ARMA-GARCH) for Stock Market Forecasting

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HONORS THESIS ABSTRACT

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ABSTRACT (100-200 WORDS):

This paper examines efficacy and limitations of time series models, namely ARIMA, GARCH, and ARMA-GARCH for stock market returns forecasting. First, the paper assesses the unique features of financial data, particularly volatility clustering and fat-tails of the return distribution, and addresses the limitations of using autoregressive integrated moving average (ARIMA) models in financial economics. Secondly, it examines the application of ARMA-GARCH models for forecasting of both conditional means as well as the conditional variance of the returns. Finally, using the standard model selection criteria such as AIC, BIC, SIC, and HQIC the forecasting performance of various candidate ARMA-GARCH models was examined. Using excess returns of MSCI World Index and excess returns from Fama-French 3-factor-model, it was found that an ARMA (1,0) + GARCH (1,1) consistently yields best results in-sample for the same period across both datasets, while showing some forecasting limitations out-of-sample.
TABLE OF CONTENTS

ABSTRACT ............................................................................................................................................. 5
INTRODUCTION AND PRIOR RESEARCH .............................................................................................. 5
MODEL SELECTION ................................................................................................................................. 12
TESTING APPROACH WITH A FAMA-FRENCH WEEKLY EXCESS RETURNS ........................................... 21
CONCLUSION ........................................................................................................................................... 26
BIBLIOGRAPHY ...................................................................................................................................... 27
APPENDIX ................................................................................................................................................ 29
1.1.1  Graph of Daily MXWO Prices ........................................................................................................... 29
1.1.2  Graph of Weekly MXWO Prices ....................................................................................................... 29
1.1.3  Graph of Weekly MXWO Returns ..................................................................................................... 30
1.1.4  Graph of Weekly MXWO Returns Based on Closing Price ............................................................... 30
1.1.5  Graph of Weekly MXWO Excess Returns ....................................................................................... 31
1.1.6  Comparison of Normal and Excess Return Distributions ................................................................. 31
1.1.7  ACF Plots of Normal, Squared, and Absolute MXWO Excess Returns ......................................... 32
1.1.8  1.1.8 PACF Plots of Normal, Squared, and Absolute MXWO Excess Returns .......................... 32
1.1.9  Box-Ljung Test of MXWO Excess Returns ..................................................................................... 33
1.1.10 Q-Q Plot of MXWO Excess Returns ............................................................................................... 33
1.3.1  Summary of ARMA (1,0) + GARCH (1,1) .................................................................................. 34
1.3.2  Summary of ARMA (1,1) + GARCH (1,1) .................................................................................. 35
1.3.3  Summary of ARMA (1,2) + GARCH (1,1) ................................................................................ 36
1.3.4  Overlap of MXWO Excess Returns and ARMA (1,0) + GARCH (1,1) ........................................ 37
1.3.5  Overlap of MXWO Excess Returns and ARMA (1,0) + GARCH (1,1) / Subset ........................... 37
1.3.6  Overlap of Squared MXWO Excess Returns and ARMA (1,0) + GARCH (1,1) ......................... 38
1.3.7  Overlap of Squared MXWO Excess Returns and ARMA (1,0) + GARCH (1,1) / Subset ......... 38
1.3.8  Overlap of Absolute MXWO Excess Returns and ARMA (1,0) + GARCH (1,1) ..................... 39
1.3.9  Overlap of Absolute MXWO Excess Returns and ARMA (1,0) + GARCH (1,1) / Subset ....... 39
1.3.10 Q-Q Plot of ARMA (1,0) + GARCH (1,1) Residuals ................................................................. 40
2.1.1  Plot of Fama-French Excess Returns / Complete Dataset ......................................................... 41
2.1.2  Plot of Fama-French Excess Returns / August 4th, 2000 – August 25th, 2017 ......................... 41
2.1.3  Comparison of MXWO and Fama-French Excess Return Distributions ................................. 42
2.3.1  Summary of ARMA (1,0) + GARCH (1,1) ............................................................................. 43
2.3.2  Summary of ARMA (1,1) + GARCH (1,1) ............................................................................. 44
2.3.3  Overlap of Fama-French Excess Returns and ARMA (1,0) + GARCH (1,1) ....................... 45
2.3.4  Overlap of Fama-French Excess Returns and ARMA (1,1) + GARCH (1,1) ................. 46
2.3.5  Overlap of Squared Fama-French Excess Returns and ARMA (1,0) + GARCH (1,1) .................. 47
2.3.6  Overlap of Squared Fama-French Excess Returns and ARMA (1,1) + GARCH (1,1) .................. 48
2.3.7  Overlap of Absolute Fama-French Excess Returns and ARMA (1,0) + GARCH (1,1) ................. 49
2.3.8  Overlap of Absolute Fama-French Excess Returns and ARMA (1,1) + GARCH (1,1) ................. 50
**ABSTRACT**

This paper examines efficacy and limitations of time series models, namely ARIMA, GARCH, and ARMA-GARCH for stock market returns forecasting. First, the paper assesses the unique features of financial data, particularly volatility clustering and fat-tails of the return distribution, and addresses the limitations of using autoregressive integrated moving average (ARIMA) models in financial economics. Secondly, it examines the application of ARMA-GARCH models for forecasting of both conditional means as well as the conditional variance of the returns. Finally, using the standard model selection criteria such as AIC, BIC, SIC, and HQIC the forecasting performance of various candidate ARMA-GARCH models was examined. Using excess returns of MSCI World Index and excess returns from Fama-French 3-factor-model, it was found that an ARMA (1,0) + GARCH (1,1) consistently yields best results in-sample for the same period across both datasets, while showing some forecasting limitations out-of-sample.

**INTRODUCTION AND PRIOR RESEARCH**

Predicting stock prices and returns is an exciting area of research given the complexity of the stock market and that the behavior of individual investors is not always rational. Researchers and investors alike have been looking for ways to maximize the profits, searching to perfect the methods to precisely forecast the movements in the stock market. This work has been partially a driving force behind a significant shift towards algorithmic trading and applying machine learning methods to investment decisions in the past decade.

With the development of the technology that enabled computation of the complex calculations in the second half of the twentieth century, the use of quantitative methods in economics and finance research has increased dramatically. In the Fifties, we see the development
of the rigorous theories that consider risk and diversification of risk in asset selection process, as well as explain risk preferences of the investors and optimal assets allocation in the portfolio under different risk aversion conditions. These theories were united in what is known as Modern Portfolio Theory and the Efficient Frontier of optimal asset allocation (Markowitz, 1952). “In this theory, an investor selects a portfolio at time t-1 that produces a stochastic return at time t. The model assumes investors are risk-averse and, when choosing among portfolios, they care only about the mean and variance of their one-period investment returns. Thus, investors choose “mean-variance-efficient portfolios” in the sense that the portfolios 1) minimize the variance of portfolio return, given expected return, and 2) maximize expected return given variance” (Fama & French, 2004).

This theory, while focusing on selecting an optimal combination of securities, outlined a critical assumption among others, today’s returns are a function of the decisions made in the past. This connectivity between the past and present actions provides researchers with an abundant amount of information contained in so-called “histories.” This leads to the idea that the “history repeats itself in that “patterns” of past price behavior will tend to recur in the future.” (Fama, 1965) However, there are also researchers that believe in “the theory of random walks which says that the future path of the price level of a security is no more predictable than the path of a series of cumulated random numbers. In statistical terms, the theory outlines that successive price changes are independent, identically distributed random variables. Most simply this implies that the series of price changes has no memory, that is, the past cannot be used to predict the future in any meaningful way.” (Fama, 1965)

In the past decades, there were several attempts made to develop forecasting models in financial economics, ranging from foreign exchange rate and interest rate forecasting to using these models to predict prices of commodities and returns on the financial assets. However, despite
successful implementation of the models in several types of research, there exists the work that does not find ARIMA models suitable for predicting returns on financial assets or exchange rates.

One of these researches includes work by Bellgard and Goldschmidt (1999) in which they used conventional techniques, including random walk, exponential smoothing, and ARIMA models to forecast exchange rates between AUD/USD. They found that the statistical forecasting precision measures do not impact profitability and foreign exchange rates directly and that the time series show nonlinear patterns that are better explained by neural network models. (Bellgard & Goldschmit, 1999)

In contrary, Awazu and Weisang (2008) used ARIMA models to forecast USD/EUR exchange rate. They found that the series of monthly USD/EUR exchange rates for the period 1994:01 to 2007:10 was best modeled by a linear relationship between the current value and its preceding three values. They also lead to the conclusion that ARIMA (1,1,1) is an adequate model for the prediction of the analyzed time series. (Awazu & Weisang, 2008)

In the recent work, “The Prediction of Exchange Rates with the Use of Auto-Regressive Integrated Moving Average Models” Spiesova (2014) confirmed that it is adequate to use the ARIMA (1,1,1) model to forecast future exchange rates on the Czech Koruna, Swedish Krona, British Pound, Polish Zloty, Hungarian Forint and the Romanian Leu vs. Euro. However, she also concluded that the ARIMA models “presented certain problems in estimating and validating the model and that those methods are more effective in the interpretation of the medium-term value.” (Spiesova, 2014)

In the way researchers disagree on whether ARIMA model is a viable method for predicting foreign exchange rates, there exists a disagreement between the researchers regarding prediction of the returns on financial assets. Partially, the cause of this disagreement is summarized in the question asked by Eugene Fama in 1965: “To what extent can the past history of a common stock's price be used to make meaningful predictions concerning the future price of the stock?”
(Fama, 1965, p. 34) This question essentially asks whether stock market prices and returns follow a random walk distribution. Fama was one of the research pioneers in this space (1965), where he used daily prices from 1957 to 1962 for stocks in DJIA (Dow-Jones Industrial Average) to examine autocorrelation coefficients. The work revealed more kurtosis (fatter tails) than that predicted from a normal distribution as well as the existence of correlations between stocks. This points to an idea that the stock market may not follow a random walk pattern, that is the past returns contain at least some information about the future returns.

Further in his paper Kon, examined and explained the unusual kurtosis (fat tails) and significant positive skewness in the distribution of daily rates of returns found by Fama for a sample of common stocks. He noted that for most of the research in financial theory, the assumption that the distribution of security rates of return be multivariate normal with parameters that are stationary over time is required. (Kon, 1984)

Lo and MacKinlay (1988) applying a test relied on variance estimators provided additional evidence regarding the Non-Random Walk evolution of the stock prices. Notably, “the random walk model is strongly rejected for the entire sample period (1962–1985) and all subperiods for a variety of aggregate returns indexes and size-sorted portfolios. Although the rejections are due largely to the behavior of small stocks, they cannot be attributed completely to the effects of infrequent trading or time-varying volatilities.” (Lo & MacKinlay, 1988)

Chang and Ting (2000) applied the methodology of Lo and MacKinlay on the weekly Taiex Index (Taiwan composite value-weighted stock market index) for the period 1971-1996 and concluded that the movements do not fit a random walk pattern. (Chang & Ting, 2000)

Unlike Chang and Ting’s (2000) study on Taiex Index, extending this study for the period 1996-2006, Lock (2007) discovered that the weekly movements of Taiex Index from 1971 to 2006 follow a Random Walk. The gap in results may be due to the nature of the market, it being in its early stages until the 1980s and later reaching maturity. (Lock, 2007)
Furthermore, research by “Tinca (2013) highlights the underlying properties of financial markets using results like conditional heavy tails, negative asymmetry, the aggregational gaussianity is more pronounced for monthly returns compared to weekly returns, volatility clustering, negative correlation between volatility and returns, positive correlation between volatility and trading volume, low significance of the mean of the daily returns. Asset pricing models tend to fail when normality assumptions are considered.” (Petrica, Stancu, & Tindeche, 2016)

Overall, in the recent research by Petrica, Stancu and Tindeche (2016), the limitation of usage of the ARIMA models in financial and monetary economics is summarized by the existence of “…fat-tails (large losses or gains are coming at a higher probability than the normal distribution would suggest) and volatility clustering - empirical properties that can’t be captured by integrated ARIMA models.” (Petrica, Stancu, & Tindeche, 2016) Additionally, they confirmed the asymmetries, sudden outbreak at irregular time intervals and periods of high and low volatility in financial time series data. They noted that “one of the most important features of the integrated ARIMA models is the assumption of constant variance, which most financial data fails to fulfill.” (Petrica, Stancu, & Tindeche, 2016)

A possible solution to the problem of rapid changes in volatility, fat tails of the distribution and clustering of volatility is the technique of applying K-mean clustering for clustering the stock market data and then using Euclidean distances for detecting the outliers introduced by Badge (2013). She explains that the “stock market data is highly chaotic and it contains a large amount of unwanted data. Detecting and removing the outliers is a fundamental problem in financial research. If the outliers are present in the data, it will give misleading results, and it also reduces the performance of prediction.” (Badge, 2013) In her work, “the stock market data passes through a multi-step process for forecasting the stock market trends. The steps are (a) normalization of stock market data (b) formation of clusters using K-mean clustering (c) finding the outliers using
Euclidean distance within the cluster and (d) applying ARIMA on clustered data. The data is then normalized using Z normalization. The attributes including open price, high price, low price, close price and trading volume are then used in the model. Clusters are then formed using K-mean clustering. K-mean clustering is a method of cluster analysis which aims to partition n observations into k clusters in which each observation belongs to the cluster with the nearest mean. The experimental result indicates that there is an improvement in the prediction result when removing outliers from the data set. In terms of mean absolute error and mean absolute percentage error, removing outliers results in significantly reduced forecasting error. K-mean clustering is an effective tool which helps to group the data in a similar pattern, while Euclidean distance helps to find the outliers within the clusters. So, for getting better forecasting results, one should reduce the effect of outliers from the financial data attributes.” (Badge, 2013)

In the past personal work, an attempt has been made to apply ARIMA model to forecast S&P 500 index. The ninety weeks of index price data between 2015:01 – 2016:11 were used to find an appropriate model. In the process, two periods of increased volatility associated with the crisis in the Chinese stock market and oil prices fluctuation respectively created roadblocks in finding a significant model until the data was adjusted for the outliers. By assigning a mean value between two neighboring points, we tried to mitigate this negative impact, which did not modify the plot of our data significantly. After the adjustments were made, an ARIMA (3,2,0) x (1,0,0) was found to fit the data quite well.
This is an example of how the clustering of volatility was affecting the results and how the adjustments for outliers helped to mitigate the problem, making Badge’s technique of K-clustering and using Euclidean Distances a viable option for this research.

From the engineering and statistical standpoint, there has also been research done in the past, attempting to develop predictive models using either artificial intelligence, statistical or hybrid approaches. One of the earlier examples of hybrid approaches is the research by Jung-Hua Wang and Jia-Yann Leu, in which “system based on a recurrent neural network was trained by using features extracted from ARIMA analysis. Empirical results showed that the networks trained using 4-year weekly data are capable of predicting up to 6 weeks market trend with acceptable accuracy.” (Wang & Leu, 1996)

Further, a hybrid model of neural networks technique and time-series models were developed by Ping-Fend Pai and Chih-Sheng Lin. In their research, they used a hybrid ARIMA and support vector machines (SVMs) to forecast stock prices. (Pai & Lin, 2005)

In one of the recent examples, a team of researchers developed a stock price predictive model solely using the ARIMA model. Through their research, they determined that the “ARIMA model has a strong potential for short-term prediction and can compete favorably with existing techniques for stock price prediction.” (Adebiyi, Adewumi, & Ayo, 2014)
MODEL SELECTION

At the beginning of the research, the goal was to find an appropriate ARIMA model that will be able to estimate and forecast stock market index in near term.

If we assume that this time series follows some ARIMA model, the conditional variance is supposed to be constant. The consistency of the conditional variance is one of the critical assumptions for predicting any future values using traditional ARMA model. When the conditional variance is not constant, varying with the past and future values, “the process is itself a random process, often referred to as the conditional variance process” (Cryer & Chan, 2014). Instead of using ARIMA model that focuses only on predicting the conditional mean of future values, the presence of the clusters of abundant volatility in financial data points out to the necessity of using the models which can simultaneously predict both the conditional mean and the conditional heteroscedasticity of the process, namely ARMA-GARCH.

As a proxy for stock market the MSCI developed countries index was selected, capturing large and mid-cap companies across 23 developed market countries, and incorporating 1,652 constituents, with roughly 85% of the free float-adjusted market capitalization in each country (MSCI, 2017). The daily index high, low, open and closing prices were collected over the period from October 1st, 2000 to October 10th, 2017. Since stocks are not traded on weekends or holidays, the data had to be calendarized accordingly and aggregated into weekly values using xts (extensive time series) package in R [Appendix 1.1.1-2]. The index values during the period show an increasing trend with a few areas of high variability. In time series analysis this points out to a non-stationary data.

To normalize the data, the prices were converted into a return metric. While usage of log returns may have theoretic and algorithmic benefits in some cases over raw returns, for convenience of the initial model selection has been done using unmodified raw returns.
\[ r_t = \frac{p_t - p_{t-1}}{p_{t-1}} \]

where \( p_t \) is the price of an asset in period \( t \), and \( p_{t-1} \) is the last period’s price. In the second portion of the paper, the analysis will be conducted using log returns for comparison. The plot of the raw returns [Appendix 1.1.3] shows an increased volatility in some time periods compared to others. This volatility is due to the financial crises, political instability, war conflicts and/or other events that lead to a rapid change in investor’s expectations. This concentration of volatility in a few time periods is called as volatility clustering in the literature. The plot also appears to be stationary with a mean of \(~0.0784\%\) making it not statistically significantly different from zero. This observation accurately follows an efficient market hypothesis, in which the expected returns should be zero eliminating a possibility of arbitrage, and suggests that a white noise model is appropriate for these data.

Looking at the nominal return rate, however, isn’t a good indicator of an overall performance of the portfolio or a financial asset. Nominal rate includes not only the actual return attributed to the performance of the asset but also returns attributed to the factors used in multi-factor models. In this research, the excess return on a security is calculated purely as the return over the risk-free rate [Appendix 1.1.5]. For purposes of calculating an excess return over the risk-free rate, the daily and weekly closing yields for 10-Year Treasury Bonds were collected. Similarly, to calculating returns on an index, the raw returns on the Treasury Bonds were calculated, corresponding to a risk-free rate on any given day or a week. The excess return data has a mean of \(~0.0899\%\), slightly higher than the unmodified returns of MXWO, but still not statistically significantly different from zero. The modification to data, however, changed the distribution of the returns [Appendix 1.1.6], increasing tails of the distribution and decreasing kurtosis from 8.136 to 5.084, and increasing skewness to -0.7055 from -0.8875.

Positive kurtosis of the distribution along with the Q-Q normal scores plot of the returns [Appendix 1.1.10] suggests a heavy-tailed distribution, with kurtosis being greater than three which is the case in normally distributed data, which is consistent the characteristics that are
prevalent in financial time series data. In simple terms, a heavy-tailed distribution represents that
the likelihood of encountering significant deviations from the mean is higher than in the case of
the normal distribution. Therefore, securities that follow this distribution have experienced returns
that have exceeded three standard deviations beyond the mean more than 0.03% of the observed
outcomes. It is interesting to note that the MXWO returns show a stronger negative tail of the
distribution.

As discussed in the introduction of this section, the remaining of the volatility clustering in
the excess returns suggests that the distribution of the returns is not independent or identical. Thus,
eliminating simple ARIMA process as a possible candidate for a predictive model for this data
series. To check the independents of the returns, I resorted to mathematical and statistical theory
that states that if the “values are truly independent, then nonlinear instantaneous transformations
such as taking logarithms, absolute values, or squaring preserves independence” (Cryer & Chan,
2014). This means that if the simple excess returns are independently and identically distributed,
so will be the absolute or squared excess returns. Thus, if there exists a significant autocorrelation
between lags, there exists evidence against the hypothesis of the independently and identically
distributed excess returns. While plotting autocorrelation (ACF) and partial autocorrelation
(PACF) functions of the excess returns already identifies evidence against independence and
identicality of the returns in the distribution, the assumption is proved further by how plotting ACF
and PACF on absolute and squared excess returns intensifies the existence of the significant lags.

To further test the autocorrelation of the squared returns, the formal Box-Ljung test can be
applied. In the absence of ARCH, “if \( m \) autocorrelations of the squared returns are used for the
test, the test statistics should be approximately chi-square distributed with \( m \) degrees of freedom”
(Cryer & Chan, 2014). Let’s assume for a moment that ARIMA model is adequate for forecasting
this time series and try to apply Box-Ljung statistics using McLeod and Li test on the data. The
test shows the significance of the test if more than two lags are included, which is consistent with
PACF and ACF of squared returns, formally showing substantial evidence for the existence of ARCH in this time-series.

The autoregressive conditional heteroskedasticity (ARCH) model was first proposed by Engle (1982) explicitly for modeling the changing variance of a time series. As was determined earlier, the returns of MXWO exhibits strong volatility clustering while maintaining a mean that is statistically non-different from zero. This suggests that that conditional variance or, as it is sometimes referred to, conditional volatility of the returns is not constant. In literature, the conditional volatility of the returns at time $t$ is typically denoted as $\sigma_{t|t-1}^2$ that is, the volatility given volatility of last period. The ARCH model, as the result, is an example of “the regression model with the conditional volatility as the response variable and the past lags of the squared returns as the covariates” (Cryer & Chan, 2014). The return series $\{r_t\}$ generated by a simple ARCH (1) model is then given by

$$r_t = \sigma_{t|t-1} \varepsilon_t$$

$$\sigma_{t|t-1}^2 = \omega + \alpha r_{t-1}^2$$

where $\alpha$ and $\omega$ are unknown parameters, $\{\varepsilon_t\}$ is a sequence of independently and identically distributed random variables each with zero mean and unit variance (known as innovations), and $\varepsilon_t$ is independent of $r_{t-j}$, $j = 1, 2, \ldots$. A main use of this type of models is to predict the future conditional variances and thus would be a great candidate for forecasting financial data with leptokurtosis.

GARCH model is considered an extension of an ARCH model. Unlike, ARCH, which involves only the most recent return, GARCH improves the accuracy of forecasting by including all the past squared returns with lesser weights corresponding to more distant volatilities. If ARCH (1) model mentioned earlier was generalized to ARCH (q) model proposed by Engle (1982) it will be represented as:
\[
\sigma^2_{t|t-1} = \omega + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2 + \cdots + \alpha_q r_{t-q}^2
\]

where \( q \) refers to the order of ARCH model. Method, proposed later by Bollerslev (1986) and Taylor (1986), included \( p \) lags of the conditional variance in the model, which was referred to as the GARCH order. The combined model called GARCH stands for generalized autoregressive conditional heteroscedasticity and follows the equation:

\[
\sigma^2_{t|t-1} = \omega + \beta_1 \sigma^2_{t-1|t-2} + \cdots + \beta_p \sigma^2_{t-p|t-p-1} + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2 + \cdots + \alpha_q r_{t-q}^2
\]

Here, \( q \) is referred to as the ARCH order, and \( p \) is referred to as GARCH order, and \( \beta \) corresponds to the weights assigned to the more distant volatilities. The combined GARCH model of order \((p,q)\) is therefore assumed, allowing for calculation of the conditional variance of the returns.

GARCH models can further be expanded in a few ways. The GARCH models assume that the conditional mean of the time series is zero, which need not always hold. The conditional variance structure of GARCH can be supplemented by a conditional mean that is modeled by some ARMA model. Specifically, let \( \{Y_t\} \) be a time series of the returns in ARIMA \((u,v)\) format:

\[
Y_t = \phi_1 Y_{t-1} + \cdots + \phi_u Y_{t-u} \theta_0 + e_t + \theta_1 e_{t-1} + \cdots + \theta_v e_{t-v}
\]

\[
e_t = \sigma_{t|t-1} e_t
\]

\[
\sigma^2_{t|t-1} = \omega + \beta_1 \sigma^2_{t-1|t-2} + \cdots + \beta_1 \sigma^2_{t-p|t-p-1} + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2
\]

The ARMA orders can be identified based on the given time series, while GARCH orders can be identified based on the squared residuals from the fitted ARMA model. Once the orders are identified, full maximum likelihood estimation for the ARMA + GARCH model can be carried out by maximizing the log-likelihood function numerically, similarly to maximizing GARCH function

\[
L(\omega, \alpha, \beta) = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^{n} \{ \log(\sigma^2_{t-1|t-2}) + \frac{r_t^2}{\sigma^2_{t-1}} \}
\]

where \( r_t \) is replaced by \( e_t \), which are recursively computed following equation above.
A common approach in statistics to quantify the goodness of fit test is the AIC (Akaike Information Criteria) statistic. To analyze all the possible combination of the models and select an appropriate candidate, a loop was created to go through all parameter combinations for both ARMA and GARCH parts that were deemed reasonable, and finally, select the model with the lowest AIC. Throughout the process, it was found that the volatility of the process can be well-modeled with GARCH (1,1) while using the process with higher parameters overestimated the levels of volatility. The conditional mean of the process was more difficult to determine that the conditional volatility and the range of parameters was set in between ARMA (0,0) & ARMA (5,5). Therefore, ARMA (0,0) & GARCH (1,1) to ARMA (5,5) & GARCH (1,1), inclusive, for each parameter pair were used to fit the model and selected the best candidate based on Akaike Information Criterion. The benefit of this approach is the ability to identify the appropriate model for various datasets quickly.

This method was applied to three modifications of MXWO excess returns from October 13th, 2000 to October 04th, 2017. Three best candidates for the model were identified, namely ARMA (1,0) + GARCH (1,1), ARMA (1,1) + GARCH (1,1), and ARMA (1,2) + GARCH (1,1) [Appendix 1.3.1-3]. Looking at the test statistics for the factors of the ARMA (1,2) + GARCH (1,1) model, the conditional mean factors of the model are shown to be statistically insignificant. This leaves ARMA (1,0) + GARCH (1,1) and ARMA (1,1) + GARCH (1,1) for further inspection.

Let’s start with comparing both models. Both models show the significance of all the terms at significance level of 5%:

<table>
<thead>
<tr>
<th>ARMA (1,0) + GARCH (1,1) Error Analysis:</th>
<th>ARMA (1,1) + GARCH (1,1) Error Analysis:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate      Std.Error  t-value  Pr(&gt;</td>
<td>t</td>
</tr>
<tr>
<td>ar1           -0.0831200  0.0378300 -2.197  0.02802</td>
<td>ar1           -0.6356000  0.2601000 -2.444  0.01452</td>
</tr>
<tr>
<td>omega         0.0000561  0.0000196 2.862  0.00421</td>
<td>omega         0.0000552  0.0000193 2.858  0.00426</td>
</tr>
<tr>
<td>alpha1        0.1018000  0.0205900 4.944  7.67E-07</td>
<td>alpha1        0.0995700  0.0198700 5.011  5.42E-07</td>
</tr>
<tr>
<td>beta1         0.8724000  0.0233100 37.428 &lt;2e-16</td>
<td>beta1         0.8748000  0.0226700 38.596 &lt;2e-16</td>
</tr>
</tbody>
</table>
Both models show appropriate results for the following tests (p > 0.05):

<table>
<thead>
<tr>
<th>Test</th>
<th>Distribution</th>
<th>ARMA (1,0) + GARCH (1,1)</th>
<th>ARMA (1,1) + GARCH (1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera</td>
<td>R</td>
<td>Chi^2 1,943.8210 0.0000</td>
<td>1,944.6690 0.0000</td>
</tr>
<tr>
<td>Shapiro-Wilk</td>
<td>R</td>
<td>W 0.9501 0.0000</td>
<td>0.9499 0.0000</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R</td>
<td>Q(10) 15.6022 0.1116</td>
<td>14.5744 0.1484</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R</td>
<td>Q(15) 19.2295 0.2035</td>
<td>18.2502 0.2497</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R</td>
<td>Q(20) 25.1398 0.1961</td>
<td>24.2152 0.2331</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R^2</td>
<td>Q(10) 5.8093 0.8310</td>
<td>5.9503 0.8194</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R^2</td>
<td>Q(15) 7.8658 0.9290</td>
<td>8.0714 0.9209</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R^2</td>
<td>Q(20) 16.1928 0.7046</td>
<td>16.0132 0.7158</td>
</tr>
<tr>
<td>LM Arch</td>
<td>R</td>
<td>TR^2 5.6685 0.9319</td>
<td>5.8235 0.9247</td>
</tr>
</tbody>
</table>

However, based on an Information Criterion Statistics the first model unanimously yields a better result. Thus, it will be explored further.

<table>
<thead>
<tr>
<th>ARMA (1,0) + GARCH (1,1)</th>
<th>ARMA (1,1) + GARCH (1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIC -3.64747</td>
<td>-3.64599</td>
</tr>
<tr>
<td>BIC -3.62586</td>
<td>-3.61897</td>
</tr>
<tr>
<td>SIC -3.64751</td>
<td>-3.64605</td>
</tr>
<tr>
<td>HQIC -3.63921</td>
<td>-3.63566</td>
</tr>
</tbody>
</table>

Overlapping the simulated returns using the ARMA (1,0) + GARCH (1,1) model (red) and the actual returns (blue) the following plot is obtained. To show the level of fit produced by a model, a zoomed-in plot for the latest 85 weeks is also obtained [Appendix 1.3.4-5].
The model gives a close fit to the actual returns both on actual data, as well as on absolute, and squared transformations [Appendix 1.3.6-9]:

As in the case of traditional ARIMA models, looking at the standardized residuals is also useful to assess the quality of the model. The normality assumption of the innovations can be explored by plotting the Q-Q normal scores plot of the residuals. If the GARCH model is correctly specified, then the standardized residuals \( \{\hat{\epsilon}_t\} \) should be close to independently and identically distributed [Appendix 1.3.10]. The residuals seem to be predominantly normally distributed with a small presence of fat tails.

The initial goal of this paper was to find a proper model that will be able to create a near-term forecast of the MXWO index returns. Given that we found a model that seems to be appropriate we can try to forecast the next ten weeks of returns. It is essential to keep in mind that after a certain point the longer lead forecasts eventually will approach the long-run variance of the model, limiting the forecasting ability to short-term (1-2 periods). Green represents a reasonable
range of forecasting; yellow represents long-term convergence to the mean.
TESTING APPROACH WITH A FAMA-FRENCH WEEKLY EXCESS RETURNS

To test this approach on a different dataset, a dataset of a 3-factor-model returns was downloaded from Kenneth French’s website. The dataset of weekly returns spans between 1926 and September 29th, 2017.

As expected, the dataset exhibits many clusters of volatility throughout the whole period. Further, to make it comparable to the previous dataset, the subset ranging from August 4th, 2000 to August 25th, 2017 was created. It appears that a new dataset has shorter tails, the lesser spread between minimum and maximum values, smaller standard deviation, increased kurtosis and decreased skewness [Appendix 2.1.3]. The mean, however, remains statistically non-significant from zero at 5% significance level with a t-statistic of 1.3064.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Min</th>
<th>Max</th>
<th>Std Div</th>
<th>Var</th>
<th>Kurtosis</th>
<th>Skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>MXWO</td>
<td>0.0008995</td>
<td>-0.27</td>
<td>0.21</td>
<td>0.04195</td>
<td>0.00176</td>
<td>5.08427</td>
<td>-0.705466</td>
</tr>
<tr>
<td>FF</td>
<td>0.0010690</td>
<td>-0.18</td>
<td>0.13</td>
<td>0.02443</td>
<td>0.00060</td>
<td>5.84469</td>
<td>-0.539177</td>
</tr>
</tbody>
</table>
The dependence of the excess returns in Fama French dataset is not expected to change from MXWO dataset. Therefore I moved on directly to fit the best model.

Following the similar approach of autofitting the ARMA-GARCH model based on AIC coefficient, yielded a suggested ARMA (4,2) + GARCH (1,1). After fitting this model to the data, it was clear that the parameters are not significant [Appendix 2.2.1]. Let’s look at two models that were close candidates for a previous dataset, namely ARMA (1,1) + GARCH (1,1) and ARMA (1,0) + GARCH (1,1) [Appendix 2.3.1-2].

Let’s start with comparing both models. Unlike the previous dataset, both models show the significance of all the terms at the significance level of 10%. With ARMA (1,0) + GARCH (1,1) being closer to a 5% confidence level cut-off.

| Parameter | Estimate | Std.Error | t-value | Pr(>|t|) |
|-----------|----------|-----------|---------|---------|
| ARMA (1,0) + GARCH (1,1) Error Analysis: | | | | |
| ar1       | -0.0597700 | 0.0365700 | -1.634 | **0.1022000** |
| omega     | 0.0000290   | 0.0000104 | 2.801  | **0.0051000** |
| alpha1    | 0.1759000   | 0.0353800 | 4.973  | **6.60E-07**  |
| beta1     | 0.7780000   | 0.0435300 | 17.873 | **< 2e-16**   |
| ARMA (1,1) + GARCH (1,1) Error Analysis: | | | | |
| ar1       | -0.6287000  | 0.2974000 | -2.114 | 0.0345000   |
| ma1       | 0.5684000   | 0.3091000 | 1.839  | **0.0659700**|
| omega     | 0.0000285   | 0.0000103 | 2.78   | 0.0054400   |
| alpha1    | 0.1727000   | 0.0352200 | 4.903  | 9.44E-07    |
| beta1     | 0.7816000   | 0.0435700 | 17.94  | **< 2e-16** |

Both models show appropriate results for the following tests (p > 0.05).

<table>
<thead>
<tr>
<th>Test</th>
<th>Distribution</th>
<th>ARMA (1,0) + GARCH (1,1)</th>
<th>ARMA (1,1) + GARCH (1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera</td>
<td>R</td>
<td>Chi^2</td>
<td>114.9453</td>
</tr>
<tr>
<td>Shapiro-Wilk</td>
<td>R</td>
<td>W</td>
<td>0.9805</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R</td>
<td>Q(10)</td>
<td>5.2341</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R</td>
<td>Q(15)</td>
<td>7.8653</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R</td>
<td>Q(20)</td>
<td>13.4778</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R^2</td>
<td>Q(10)</td>
<td>10.1249</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R^2</td>
<td>Q(15)</td>
<td>11.9476</td>
</tr>
<tr>
<td>Ljung-Box</td>
<td>R^2</td>
<td>Q(20)</td>
<td>14.0112</td>
</tr>
<tr>
<td>LM Arch</td>
<td>R</td>
<td>TR^2</td>
<td>9.9173</td>
</tr>
</tbody>
</table>

22
However, based on an Information Criterion Statistics the first model unanimously yields a better result. Because two models are significant at different confidence levels, I will attempt to analyze both.

<table>
<thead>
<tr>
<th>Model</th>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARMA (1,0) + GARCH (1,1)</td>
<td>-4.85100</td>
<td>-4.82949</td>
<td>-4.85104</td>
<td>-4.84278</td>
</tr>
<tr>
<td>ARMA (1,1) + GARCH (1,1)</td>
<td>-4.85003</td>
<td>-4.82313</td>
<td>-4.85009</td>
<td>-4.83975</td>
</tr>
</tbody>
</table>

Overlapping the simulated returns using the ARMA (1,0) + GARCH (1,1) model (red) and the actual returns (blue) the following plot is obtained. To show the level of fit produced by a model, a zoomed-in plot for the latest 85 weeks is also obtained [Appendix 2.2.3-4].
Both models seem to have a close fit to the actual data on actual returns. Let’s look closer at squared and absolute return transformations [Appendix 2.3.5-8]:

![Graphs showing overlap of squared and absolute returns with fitted models.](image-url)
It appears that ARMA (1,0) + GARCH (1,1), yet again, exhibits closer fit to the actual data without overfitting it or showing increased volatility.

Using this model to predict next five periods of returns we get the following. Green shadowing represents a reasonable range of forecasting, yellow shadowing represents long-term convergence to the mean, limiting the forecasting ability to short-term (1-2 periods). Green line represents out-of-sample predicted returns, red line represents actual returns, and blue represents the in-sample fit of the model.

<table>
<thead>
<tr>
<th>t</th>
<th>Predicted</th>
<th>Actual</th>
<th>Actual - Predicted</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>-0.030%</td>
<td>0.090%</td>
<td>-1.520%</td>
</tr>
<tr>
<td>Predicted</td>
<td>-0.030%</td>
<td>0.090%</td>
<td>-1.520%</td>
</tr>
<tr>
<td>Actual - Predicted</td>
<td>0.000%</td>
<td>0.000%</td>
<td>0.000%</td>
</tr>
</tbody>
</table>

While the model exhibited a robust in-sample fit, out of sample, it seems to underestimate both positive and negative returns. In case of 2017-09-01 actual return of 1.61% model predicted the return of 0.49% with the delta of -1.12%. In case of 2017-09-08 return of -0.77%, the model predicted the return of -0.029%, with the delta of .0741%. Thus, if we used it to invest, we would
have missed out on -1.12% of returns on September 1\textsuperscript{st} and generated .741\% return on September 8\textsuperscript{th} over market return.

\textbf{CONCLUSION}

The variation between predicted and actual returns shows the difficulty in stock market returns forecasting out-of-sample even with the model that has an excellent in-sample fit. This may be due to the weekly nature of returns. Using weekly returns aggregates daily returns in which case a lot of the information about variation and the momentum is lost. A model fitted to daily returns or lesser time periods arguably may yield better forecasting ability in near term. This can be further improved by incorporating the ability to adjust for the newly available data automatically.


APPENDIX

1.1.1 Graph of Daily MXWO Prices

1.1.2 Graph of Weekly MXWO Prices
1.1.3 Graph of Weekly MXWO Returns

1.1.4 Graph of Weekly MXWO Returns Based on Closing Price
1.1.5 Graph of Weekly MXWO Excess Returns

Weekly Excess Return of MXWO

1.1.6 Comparison of Normal and Excess Return Distributions

Histogram of Weekly Returns

Histogram of Excess Weekly Returns

Overlapping Histogram
1.1.7 ACF Plots of Normal, Squared, and Absolute MXWO Excess Returns

1.1.8 PACF Plots of Normal, Squared, and Absolute MXWO Excess Returns
1.1.9 Box-Ljung Test of MXWO Excess Returns

1.1.10 Q-Q Plot of MXWO Excess Returns
1.3.1 Summary of ARMA (1, 0) + GARCH (1, 1)

Title:
GARCH Modelling

Call:
garchFit(formula = ~arma(1, 0) + garch(1, 1), data = MXWO.Rf,
include.mean = FALSE)

Mean and Variance Equation:
data ~ arma(1, 0) + garch(1, 1)
<environment: 0x0000000020f37120>
[data = MXWO.Rf]

Conditional Distribution:
norm

Coefficient(s):
ar1 omega alpha1 betal
-8.3122e-02 5.6068e-05 1.0177e-01 8.7235e-01

Std. Errors:
based on Hessian

Error Analysis:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
ar1       | -8.312e-02 | 3.783e-02 | -2.197  | 0.02802 * |
| omega    | 5.607e-05  | 1.959e-05 | 2.862   | 0.00421 **|
| alpha1   | 1.018e-01  | 2.059e-02 | 4.944   | 7.67e-07 ***|
| betal    | 8.724e-01  | 2.331e-02 | 37.428  | < 2e-16 ***|

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
1619.829 normalized: 1.828249

Description:
Sat Nov 25 19:20:14 2017 by user: olegg

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>1943.821</td>
<td>0</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>0.9501352</td>
<td>0</td>
</tr>
<tr>
<td>Ljung-Box Test (10)</td>
<td>15.60217</td>
<td>0.116012</td>
</tr>
<tr>
<td>Ljung-Box Test (15)</td>
<td>19.22953</td>
<td>0.203526</td>
</tr>
<tr>
<td>Ljung-Box Test (20)</td>
<td>25.13976</td>
<td>0.1961364</td>
</tr>
<tr>
<td>Ljung-Box Test (10)</td>
<td>5.809323</td>
<td>0.8310205</td>
</tr>
<tr>
<td>Ljung-Box Test (15)</td>
<td>7.865819</td>
<td>0.9290494</td>
</tr>
<tr>
<td>Ljung-Box Test (20)</td>
<td>16.1928</td>
<td>0.7045932</td>
</tr>
<tr>
<td>LM Arch Test (10)</td>
<td>5.668502</td>
<td>0.9318623</td>
</tr>
</tbody>
</table>

Information Criterion Statistics:

<table>
<thead>
<tr>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.647469</td>
<td>-3.625859</td>
<td>-3.647510</td>
<td>-3.639208</td>
</tr>
</tbody>
</table>
1.3.2 Summary of ARMA (1,1) + GARCH (1,1)

Title:
GARCH Modelling

Call:
garchFit(formula = ~arma(1, 1) + garch(1, 1), data = MXWO.Rf,
include.mean = FALSE)

Mean and Variance Equation:
data ~ arma(1, 1) + garch(1, 1)
<environment: 0x0000000028c07ef0>
[data = MXWO.Rf]

Conditional Distribution:
norm

Coefficient(s):
ar1      ma1      omega     alpha1     beta1
-6.3563e-01 5.6985e-01 5.5146e-05 9.9568e-02 8.7478e-01

Std. Errors:

Error Analysis:

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
1620.172    normalized: 1.828636

Description:
Sat Nov 25 19:19:59 2017 by user: olegg

Standardised Residuals Tests:

Information Criterion Statistics:
AIC       BIC       SIC       HQIC
-3.645986  -3.618973  -3.646049  -3.635659
1.3.3 Summary of ARMA (1,2) + GARCH (1,1)

Title:
GARCH Modelling

Call:
garchFit(formula = ~arma(1, 2) + garch(1, 1), data = MXWO.Rf, 
include.mean = FALSE)

Mean and Variance Equation:
data ~ arma(1, 2) + garch(1, 1)
<environment: 0x0000000028766ff0>
[data = MXWO.Rf]

Conditional Distribution:
norm

Coefficient(s):
ar1  ma1  ma2  omega  alpha1  beta1
4.4360e-01 -5.2667e-01 6.8759e-02 5.5253e-02 1.0031e-01 8.7400e-01

Std. Errors:
based on Hessian

Error Analysis:
Estimate Std. Error t value Pr(>|t|)
ar1  4.436e-01  3.855e-01 1.151  0.24987
ma1 -5.267e-01 3.854e-01 -1.366 0.17179
ma2  6.876e-02 4.962e-02 1.386  0.16587
omega 5.525e-05  1.941e-05 2.846 0.00443 **
alpha1 1.003e-01 2.034e-02 4.932 8.16e-07 ***
beta1  8.740e-01  2.310e-02 37.830 < 2e-16 ***

---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
1620.217  normalized: 1.828687

Description:
Sat Nov 25 19:20:24 2017 by user: olegg

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R^2 Chi² 1894.415 0</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>R W 0.9509406 0</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(10) 14.38043 0.1563352</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(15) 18.07575 0.2587043</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(20) 23.98481 0.2430562</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(10) 6.046879 0.813094</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(15) 8.181371 0.9163055</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(20) 16.04914 0.7135713</td>
</tr>
<tr>
<td>LM Arch Test</td>
<td>R TR^2 5.92573 0.9197795</td>
</tr>
</tbody>
</table>

Information Criterion Statistics:

<table>
<thead>
<tr>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.643831</td>
<td>-3.611415</td>
<td>-3.643922</td>
<td>-3.631439</td>
</tr>
</tbody>
</table>
1.3.4 Overlap of MXWO Excess Returns and ARMA (1,0) + GARCH (1,1)

1.3.5 Overlap of MXWO Excess Returns and ARMA (1,0) + GARCH (1,1) / Subset
1.3.6 Overlap of Squared MXWO Excess Returns and ARMA (1,0) + GACH (1,1)

1.3.7 Overlap of Squared MXWO Excess Returns and ARMA (1,0) + GACH (1,1) / Subset
1.3.8 Overlap of Absolute MXWO Excess Returns and ARMA (1,0) + GACH (1,1)

Overlap of Absolute Returns and Fitted ARMA (1,0) + GARCH (1,1)

1.3.9 Overlap of Absolute MXWO Excess Returns and ARMA (1,0) + GACH (1,1) / Subset

Overlap of Absolute Returns and Fitted ARMA (1,0) + GARCH (1,1)
1.3.10 Q-Q Plot of ARMA (1,0) + GACH (1,1) Residuals
2.1.1  Plot of Fama-French Excess Returns / Complete Dataset

FF Excess Returns [July 20th, 1926 - Sept 29th, 2017]

2.1.2  Plot of Fama-French Excess Returns / August 4th, 2000 – August 25th, 2017

2.1.3 Comparison of MXWO and Fama-French Excess Return Distributions
2.3.1 Summary of ARMA (1,0) + GARCH (1,1)

Call:
garchFit(formula = ~arma(1, 0) + garch(1, 1), data = FF.Rf, include.mean = FALSE)

Mean and Variance Equation:
data ~ arma(1, 0) + garch(1, 1)
<environment: 0x0000000032fdbac0>[data = FF.Rf]

Conditional Distribution:
norm

Coefficient(s):

<table>
<thead>
<tr>
<th>ar1</th>
<th>omega</th>
<th>alphal</th>
<th>betal</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5.9768e-02</td>
<td>2.9036e-05</td>
<td>1.7593e-01</td>
<td>7.7800e-01</td>
</tr>
</tbody>
</table>

Std. Errors:

based on Hessian

Error Analysis:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|---------|
| ar1      | -5.977e-02 | 3.657e-02 | -1.634 | 0.1022 |
| omega    | 2.904e-05  | 1.037e-05 | 2.801  | 0.0051 **|
| alphal   | 1.759e-01  | 3.538e-02 | 4.973  | 6.6e-07 ***|
| betal    | 7.780e-01  | 4.353e-02 | 17.873 | <2e-16 ***|

---

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:

2165.121 normalized: 2.42999

Description:
Sun Dec 03 17:20:58 2017 by user: olegg

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p-value</th>
</tr>
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<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R Chi^2</td>
<td>114.9453 0</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>R W</td>
<td>0.9805131 1.566897e-09</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(10)</td>
<td>5.234145 0.874998</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(15)</td>
<td>7.86534 0.9290677</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(20)</td>
<td>13.47775 0.8559586</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(10)</td>
<td>10.1249 0.4296048</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(15)</td>
<td>11.94764 0.6829883</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(20)</td>
<td>14.01121 0.8299271</td>
</tr>
<tr>
<td>LM Arch Test</td>
<td>R TR^2</td>
<td>9.917257 0.6232196</td>
</tr>
</tbody>
</table>

Information Criterion Statistics:

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<tr>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
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<tbody>
<tr>
<td>-4.851002</td>
<td>-4.829488</td>
<td>-4.851042</td>
<td>-4.842779</td>
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</tbody>
</table>
2.3.2 Summary of ARMA (1,1) + GARCH (1,1)

Call:
garchFit(formula = ~arma(1, 1) + garch(1, 1), data = FF.Rf, include.mean = FALSE)

Mean and Variance Equation:
data ~ arma(1, 1) + garch(1, 1)
<environment: 0x0000000020838298>
[data = FF.Rf]

Conditional Distribution:
  norm

Coefficient(s):
  ar1  ma1  omega  alpha1  beta1
-6.2866e-01 5.6838e-01 2.8506e-05 1.7270e-01 7.8157e-01

Std. Errors:
  based on Hessian

Error Analysis:
  Estimate  Std. Error  t value  Pr(>|t|)
  ar1      -6.287e-01      2.974e-01     -2.114 0.03450 *
  ma1       5.684e-01      3.091e-01      1.839 0.06597 .
  omega    2.851e-05      1.025e-05      2.780 0.00544 **
  alpha1   1.727e-01      3.522e-02      4.903 9.44e-07 ***
  beta1    7.816e-01      4.357e-02     17.940 < 2e-16 ***

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
  2165.687 normalized: 2.430625

Description:
  Sun Dec  3 17:20:41 2017 by user: olegg

Standardised Residuals Tests:

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<tr>
<th>Test</th>
<th>Statistic</th>
<th>p-Value</th>
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</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R Chi^2</td>
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<td>Shapiro-Wilk Test</td>
<td>R W</td>
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Information Criterion Statistics:

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<th>HQIC</th>
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<tbody>
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<td>-4.850090</td>
<td>-4.839749</td>
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2.3.3 Overlap of Fama-French Excess Returns and ARMA (1,0) + GARCH (1,1)
2.3.4 Overlap of Fama-French Excess Returns and ARMA (1,1) + GARCH (1,1)
2.3.5 Overlap of Squared Fama-French Excess Returns and ARMA (1,0) + GARCH (1,1)

Overlap of Squared FF Returns and Fitted ARMA (1,0) + GARCH (1,1)

Overlap of Squared FF Returns and Fitted ARMA (1,0) + GARCH (1,1)
2.3.6 Overlap of Squared Fama-French Excess Returns and ARMA (1,1) + GARCH (1,1)

Overlap of Squared FF Returns and Fitted ARMA (1,1) + GARCH (1,1)

Overlap of Squared FF Returns and Fitted ARMA (1,1) + GARCH (1,1)
2.3.7 Overlap of Absolute Fama-French Excess Returns and ARMA (1,0) + GARCH (1,1)
2.3.8 Overlap of Absolute Fama-French Excess Returns and ARMA (1,1) + GARCH (1,1)