Qualitative Theory of Differential Equations

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The problem of stability is of primary concern in the qualitative theory of differential equations, and has occupied mathematicians for the past century. The problem appears when considering solutions to the differential equation $\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x})$ where $\mathbf{x}=(x_1(t),\ldots,x_n(t))^T$ and $\mathbf{f}(t,\mathbf{x})$ is a nonlinear function of $x_1,\ldots,x_n$. While no known method of solving this equation explicitly exists even for the case n=2, it is possible to discuss the qualitative properties of $x_1(t)$ and $x_2(t)$ where $x_1(t)$ and $x_2(t)$ denote, for example, the populations, at time t, of two competing species. The qualitative attributes under consideration include points of equilibrium and the question of the stability of solutions in neighborhoods of these points. Note that an equilibrium point occurs when the values z_1 and z_2 exist at which the two species can coexist together in a steady state; that is, if there are z_1 and z_2 such that $x_1(t)=z_1$ and $x_2(t)=z_2$ is a solution of the differential equation (above), then (z_1,z_2) is a point of equilibrium. Further note that the problem of stability arises when considering what will happen when members of species 1 are added to the picture. The question is whether the values $x_1(t)$ and $x_2(t)$ will remain near their equilibrium values as time t tends to infinity, or if species 1 will gain a significant advantage from the added members and commence eliminating species 2. While points of equilibrium can be determined directly by observing that $\dot{\mathbf{x}}(t)\equiv 0$ if $\mathbf{x}(t)\equiv x^0$ and therefore, x^0 is an equilibrium value of the differential equation, if, and only if, $\mathbf{f}(t,x^0)\equiv 0$, the question of stability is not as easily resolved. That is, because it is not possible to solve the diffential equation explicitly, the only case of stability it is feasible to consider solving, is the situation where $\mathbf{f}(t,\mathbf{x})$ does not depend on t explicitly, but is a function of $\mathbf{x}$ alone; differential equations such as these are labeled "autonomous." Complete resolution of the stability question is only possible (generally) for two cases of autonomous differential equations, namely: linear systems and equilibrium solutions.

Every solution of the linear differential equation $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ can be identified as either stable or unstable; that is, the question of stability can be answered conclusively. To clarify stability, let $\mathbf{x}=\phi(t)$ be a solution to the differential equation $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, and consider the following formal definition of stability:

> The solution $\mathbf{x}=\phi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ is stable if every solution $\psi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ which starts sufficiently close to $\phi(t)$ at $t=0$ must remain close to $\phi(t)$ for all future time t. The solution $\phi(t)$ is unstable if there exists at least one solution $\psi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ which starts near $\phi(t)$ at $t=0$ but which does not remain close to $\phi(t)$ for all future time. More precisely, the solution $\phi(t)$ is stable if for every $\epsilon>0$ there exists $\delta=\delta(\epsilon)$ such that $|\psi_j(t)-\phi_j(t)|<\epsilon$ if $|\psi_j(0)-\phi_j(0)|<\delta(\epsilon)$, $j=1,\ldots,n$ for every solution $\psi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$.[1]

It is now possible to settle the stability question for $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ using the following important theorem:

> Theorem 1. (a) Every solution $\mathbf{x}=\phi(t)$ of $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ is stable if all the eigenvalues of $\mathbf{A}$ have negative real part.
> (b) Every solution $\mathbf{x}=\phi(t)$ of $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ is unstable if at least one eigenvalue of $\mathbf{A}$ has positive real part.
> (c) Suppose that all the eigenvalues of $\mathbf{A}$ have real part ≤ 0 and $\lambda_1=i\sigma_1,\ldots,\lambda_1=i\sigma_1$

have zero real part. Let $\lambda_j = i\sigma_j$ have multiplicity k_j. This means that the characteristic polynomial of $\mathbf{A}$ can be factored into the form

$$p(\lambda) = (\lambda - i\sigma_1)^{k_1} \ldots (\lambda - i\sigma_l)^{k_l} q(\lambda)$$

where all the roots of $q(\lambda)$ have negative real part. Then, every solution $\mathbf{x} = \phi(t)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable if $\mathbf{A}$ has k_j linearly independent eigenvectors for each eigenvalue $\lambda_j = i\sigma_j$. Otherwise, every solution $\phi(t)$ is unstable.[2]

While proving part (a) of Theorem 1, a phenomenon entitled asymptotic stability can be noted. That is, if all the eigenvalues of $\mathbf{A}$ have negative real part, then every solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ tends to zero as t tends to infinity; hence, not only is the equilibrium solution $\mathbf{x}(t) \equiv 0$ stable, but each solution $\phi(t)$ of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ nears it as t tends to infinity; label this phenomenon asymptotic stability and note further that it is a very strong type of stability. Applying Theorem 1 to answer the question of stability for linear systems requires employing knowledge gained in MATH 240 (Linear Algebra) with respect to finding characteristic polynomials and eigenvalues, as demonstrated by the worked problems in Appendix I: Problem Portfolio.

The problem of stability can also be resolved for other solutions, where the equation to be considered is: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$ where $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ is very small when compared with $\mathbf{x}$. Determining stability for these solutions is accomplished by employing a parallel version of Theorem 1 as follows:

Theorem 2. Suppose that the vector-valued function $\mathbf{g}(\mathbf{x})/\|\mathbf{x}\| \equiv \mathbf{g}(\mathbf{x})/\max\{|x_1|, \ldots, |x_n|\}$ is a continuous function of $x_1, \ldots, x_n$ which vanishes for $\mathbf{x} = 0$. Then,

(a) The equilibrium solution $\mathbf{x}(t) \equiv 0$ of

$\dot{x}=Ax+g(x)$ is asymptotically stable if the equilibrium solution $x(t)\equiv 0$ of the "linearized" equation $\dot{x}=Ax$ is asymptotically stable. Equivalently, the solution $x(t)\equiv 0$ of $\dot{x}=Ax+g(x)$ is asymptotically stable if all the eigenvalues of A have negative real part.
(b) The equilibrium solution $x(t)\equiv 0$ of $\dot{x}=Ax+g(x)$ is unstable if at least one eigenvalue of A has positive real part.
(c) The stability of the equilibrium solution $x(t)\equiv 0$ of $\dot{x}=Ax+g(x)$ cannot be determined from the stability of the equilibrium solution $x(t)\equiv 0$ of $\dot{x}=Ax$ if all the eigenvalues of A have real part ≤ 0 but at least one eigenvalue of A has zero real part.[3]

The importance of Theorem 2 becomes apparent, since it can be used to determine the stability of equilibrium solutions of arbitrary autonomous differential equations. Consider x^0, an equilibrium value of the differential equation $\dot{x}=f(x)$, and set $z(t)=x(t)-x^0$, then $\dot{z}=\dot{x}=f(x^0+z)$. Furthermore, $z(t)\equiv 0$ is an equilibrium solution of $\dot{z}=\dot{x}=f(x^0+z)$, the stability of $z(t)\equiv 0$ is equivalent to the stability of $x(t)\equiv x^0$, and finally, $f(x^0+z)$ can also be written as, $f(x^0+z)=Az+g(z)$ where $g(z)$ is small when compared to z. Based upon these facts, the following lemma can be presented:

Lemma 1. Let $f(x)$ have two continuous partial derivatives with respect to each of its variables $x_1,\ldots,x_n$. Then, $f(x^0+z)$ can be written in the form

$$f(x^0+z)=f(x^0)+Az+g(z)$$

where $g(z)/\max\{|z_1|,\ldots,|z_n|\}$ is a continuous function of z which vanishes for $z=0$.[4]

Theorem 2 and Lemma 1 yield the three-step algorithm for deciding whether an equilibrium solution $x(t)\equiv x^0$ of $\dot{x}=f(x)$ is stable or unstable:

1. Set $z=x-x^0$.
2. Write $f(x^0+z)$ in the form $Az+g(z)$ is a vector-valued polynomial in $z_1,\ldots,z_n$ beginning with terms of order two or more.

3. Compute the eigenvalues of A. If all the eigenvalues of A have negative real part, then $\mathbf{x}(t)\equiv\mathbf{x}^0$ is asymptotically stable. If one eigenvalue of A has positive real part, then $\mathbf{x}(t)\equiv\mathbf{x}^0$ is unstable.[5]

Examples of finding all the equilibrium solutions of a system of equations and of determining the stability or instability of these solutions, can be found in Appendix I.

Another facet of qualitative theory is the geometric study of differential equations. Since the intent of pursuing qualitative theory is to acquire the most complete description possible of all solutions of the system of differential equations $dx/dt=f(x,y)$, $dy/dt=g(x,y)$, the graphs of the solutions of the system would assist the qualitative study. Observe that each solution $x=x(t)$, $y=y(t)$ of $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ traces out a curve in 3-space (t,x,y). Note that each of the solutions $x=x(t)$, $y=y(t)$ where $t_0\le t\le t_1$ of $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ also traces out a curve in 2-space (x,y), or the x-y plane. That is, the set of points $(x(t),y(t))$ define a curve C in the x-y plane, as t traverses the interval (t_0,t_1). The curve thus defined is called the orbit or trajectory of the solution $x=x(t)$, $y=y(t)$ and the x-y plane is called the phase-plane of the solutions. While graphing the solutions in 3-space is possible (ex. helix), it is not as manageable as considering the phase-plane. Defining the curve C in 2-space allows for enough diversity to satisfy the intent of qualitative theory, yet does not involve the complexity of three-dimensional plotting. Some examples of 2-dimensional orbits are: spirals, lines, parabolas, sets of curves, and families of elipses. Note that the orbits of the solutions $x=x(t)$, $y=y(t)$ of

$dx/dt=f(x,y), dy/dt=g(x,y)$ are the solution curves of the first-order scalar equation $dy/dx=g(x,y)/f(x,y)$, which, unfortunately, cannot (in general) be solved explicitly. However, securing a precise representation of all the orbits of $dx/dt=g(x,y), dy/dt=g(x,y)$ can still be accomplished, since the system of differential equations $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ sets up a direction field in the x-y plane. That is, the system of differential equations $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ indicates the velocity at which a solution moves along its orbit, as well as which direction the solution is moving.. Furthermore, the concept of an orbit can be extended to include curves in three-dimensional space, which is labeled the phase-spaces of the solutions. One of the advantages of considering the orbit or trajectory of the solution, rather than the solution itself, is that it is often feasible to produce the orbit of a solution, without prior knowledge of the solution.

Several important applications of the qualitative theory of differential equations are collectively entitled "mathematical theories of war," and include, L. F. Richardson's theory of conflict, as well as Langhester's combat models and the battle of Iwo Jima. Beginning with Richardson's theory, a mathematical model is constructed which delineates the relationship between two nations, each of which is resolved to defend itself from possible attack by the other. Each of the two nations believes the danger of attack to be quite real, and bases its apprehensions upon the preparedness of the other to engage in war. The model under consideration

(Richardson's) is $dx/dt=ky-ax+g$, $dy/dt=lx-by+h$, where $x(t)$ denotes the war potential (armaments) of nation one, $y(t)$ specifies the war potential of nation two, and the rate of change of each depends upon the constants a,b,g,h,k, and l. Note that k indicates the war readiness of nation two, l denotes the war readiness of nation one, a represents the cost of armaments for nation one, b depicts expense of arming nation two, and, finally, g and h designate the grievances that nation one feels toward nation two, and that nation two holds against nation one, respectively. By comparing the European arms race of 1909-1914 (World War I), with Richardson's model, and by incorporationg data such as the defense budgets of each alliance, a high correllation is found between the two, with regard to Richardson's choice of constant values. Understanding of the arms race models as a group was expanded and deepened through this study, as well as through studious endeavors in MATH 360 (Model Building) which included work on the competitive hunter's model, the predator-prey model, and Richardson's model. Lanchestrian models extended those studies to include models for conventional guerilla combat, for example, $dx/dt=-cxy+f(t)$, $dy/dt=-dx+g(t)$, where x denotes guerilla forces, c indicates the combat effectiveness coefficient of the opponent y, $cx(t)y(t)$ designates the combat loss rate for the guerilla force (x) and $dx(t)$ characterizes the loss rate of the non-guerilla force, and finally, $f(t)$ and $g(t)$ express the re-inforcement rates of x and y, respectively. Lanchester's work also produced a mathematical model for a conventional