NORTHERN ILLINOIS UNIVERSITY

Equitable Mathematics Education Through Pursuing Quantitative Reasoning

A Thesis Submitted to the

University Honors Program

In Partial Fulfillment of the

Requirements of the Baccalaureate Degree

With Upper Division Honors

Department Of

Mathematical Sciences

By

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DeKalb, Illinois

December 2013

University Honors Program

Capstone Approval Page

Capstone Title (print or type)

Equitable Mathematics Education Through Pursuing Quantitative Reasoning

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Date of Approval (print or type) 5/10/2013

HONORS THESIS ABSTRACT
THESIS SUBMISSION FORM

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THESIS TITLE: Equitable Mathematics Education Through Pursuing Quantitative Reasoning

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ADVISOR'S DEPARTMENT: Department of Mathematical Sciences

DISCIPLINE: Mathematics Education YEAR: Senior

PAGE LENGTH: BIBLIOGRAPHY: Included

ILLUSTRATED: N/A PUBLISHED: No

LIST PUBLICATION: N/A COPIES AVAILABLE: Hard Copy

ABSTRACT (100-200 WORDS): The purpose of this project is to develop means of addressing discrepancies between the mathematics education of students from different socioeconomic backgrounds. The central question of the project is: How can I as a teacher foster quantitative reasoning so that underrepresented students gain access to meaningful understanding of mathematics? Access to mathematics is important in order to gain access to economic opportunities, meaning it is essentially a social justice issue. The project entailed secondary research of existing paradigms and research projects. The major finding of the project is that some of the classroom causes of inequity in mathematics education can be addressed by implementing quantitative reasoning in the classroom. Quantitative reasoning does not address all issues of equity, but it does address the need to provide students with means of sense-making and thus access to meaningful understanding. To implement the findings of the secondary research, I developed lesson plans geared to improve student access to mathematical meaning through quantitative reasoning. The lesson plan developed a quantitatively meaningful approach to studying rational functions.

Introduction

Origin of Capstone Topic

My preparation for teaching thus far exposed me to the disparity between the mathematics instruction available to students of different socioeconomic backgrounds. I wanted to explore how the tools of mathematics instruction I learned in my program could help address some of these disparities at the level of the individual classroom. As an extension of a course, MATH 416, I took I determined a possible way to address these disparities is through instruction geared toward fostering quantitative reasoning.

I drew upon my clinical experiences in the mathematics education program and my participation in the Golden Apple teaching fellow program to identify my focus of research. I tutored at Chicago Public schools during summer school and worked with students from Aurora at a Math and Science Camp sponsored by IMSA. These experiences differed drastically from each other in terms of socioeconomic status and mathematics preparation. I perceived that there was a strong link between these students backgrounds socio-culturally and their access to mathematical understanding. I saw that schools in Chicago did not have the same resources we were provided while working with the students in Aurora. I also noticed that students in the summer school were much further behind the students I worked with in Aurora. For example, I worked with third grade students who were taking summer school in order to move on to the fourth grade. I had to teach them how to add, subtract, multiply, and divide, which are skills that should have been mastered at this point. Those students missed out on discussing more advanced mathematical concepts. Other Golden Apple scholars told me that they were teaching sixth grade students those same concepts. By contrast in Aurora, I taught students who had just finished sixth and seventh grade about exponential growth through the spread of a disease. We were able to model the behavior of the disease by using graphing calculators, and discuss the meaning of the model. I saw an obvious discrepancy in the two populations' mathematical preparation.

My experiences at Golden Apple led me to the idea that I would like to address the extreme differences I saw in the mathematical educations of the two populations I taught over the summers. I wanted to be able to provide mathematical education to both populations that provided them with the opportunity to gain a deeper mathematical understanding. It is not fair that the students in Chicago were not provided the same opportunities to learn and grow in their mathematical knowledge as the students in Aurora in large part due to their circumstances. As mentioned previously the students during summer school at the Chicago Public schools were underachieving, and the students at the Math and Science Camp in Aurora were overachieving. The methods of instruction differed extremely from the two locations. In Chicago, we practiced mathematical concepts through repetition and contextless exercises. On the other hand in Aurora, we selected real-world problems in order to explore and learn a mathematical concept. From these experiences, I believe that students understand mathematical concepts more readily when the latter teaching method is implemented. The method of using context to teach mathematical concepts involves quantitative reasoning, therefore, I determined that we could look at the idea of making mathematics education more equitable across different socioeconomic statues through the use of quantitative reasoning in the classroom. To be clear, I am aware there are students in Chicago that do excel in mathematics, and students in Aurora that are underachieving in mathematics. I used this example from my experiences to demonstrate that an inequality in mathematics education exists for students.

Relevance of Capstone Topic

I believe my experiences of teaching in Chicago and Aurora demonstrated the need to address the lack of equitable mathematics education. This topic should be relevant for any mathematics teacher due to the Equity Principle established by the National Council of Teachers of Mathematics (NCTM). The Equity Principle states: "Excellence in mathematics education requires equity – high expectations and strong support for all students" (NCTM). I, therefore, believe that determining a means, such as quantitative reasoning, of providing equitable mathematics education students to will help me to achieve NCTM's Equity Principle in my classroom. Another concern of making mathematics education equitable is that depending on the location of a school district, resources for their students may be scarce. It is widely acknowledged and politicized that American Mathematics education exhibits definitive achievement gaps between different socioeconomic subsets of students, even within the same school. I perceive that the achievement gap reveals differences between student's mathematical educations rather than between students, but I will later go into more detail about how to address the achievement gap and what it means in the classroom. Students differing mathematical educations should be of a concern to mathematics teachers because "mathematics acts as a 'gatekeeper' to economic success, active citizenship, and higher education in our society" (Wager, 83). By providing students with "access" to mathematics, student will gain "access" to future economic opportunities making mathematics education a social justice issue. Through the use of quantitative reasoning, I hope that I may be able to make mathematics education more equitable for my students.

Central Question of Capstone

My capstone's central question is: How can I as a teacher foster quantitative reasoning so that underrepresented students gain access to meaningful understanding of mathematics? In later sections I will define 'quantitative reasoning' and the terminology associated with it. I will also define the 'access' and how it relates to equity. I will then provide my operational definition for 'meaningful understanding' and how it relates to particular mathematical concepts. Once my question is completely defined, I will then explore my question further.

Goals of Capstone

As mentioned before, I wanted to have a topic that would help me become a better teacher, and a topic that would be applicable to my future mathematics classroom. Another goal of this capstone is to define what it means for mathematics education to be equitable and finding a means to achieve equitable mathematics education. This involved researching the causes of inequity in mathematics education both inside and outside the classroom. I then examine why quantitative reasoning is an appropriate means of making mathematical education equitable relative to my stated goals for student learning. The next goal is to research quantitative reasoning, so that I understand how to implement quantitative reasoning tasks in my classroom. I will also practice implementing quantitative reasoning by adapting a lesson plan for teaching rational functions in context.

Definitions-Background Literature

Defining "Equity" and "Access"

Connecting mathematics learning to economic opportunity justifies why the Equity Principle is one of the NCTM's six guiding principles for mathematics education. Due to the ambiguity of the previous statement, the principle still does not help in defining what exactly equity is. To help further define equity, I will use the three major components of the Equity Principle.

The first component of the Equity Principle is: "Equity requires high expectations and worthwhile opportunities for all" (The Equity Principle). Contrary to many people's beliefs, students that are underachieving in mathematics should not have the expectations lowered for them. By lowering expectations for students, teachers are providing multiple negative messages to the students. The teacher is first signaling that underachieving in mathematics is acceptable. The teacher also signals to the students that the teacher does not believe that they are capable of learning mathematics at a high level. When a teacher or teachers repeatedly put the notion inside a student's head that he or she is not capable of participating in mathematics, then the teacher is playing right into developing an identity of helplessness. Teachers should instead be developing an identity of "*knowers* and *doers* of mathematics" (Silver). When teachers hold all students to high expectations, then the students will be more likely to respond in a positive manner in an attempt to reach those high expectations rather than believing not being able to do mathematics is acceptable.

Having high expectations for all students leads to providing worthwhile opportunities for all. By having high expectations for all students, teachers are already signaling to the students that students have the ability to participate and learn mathematics. Another aspect that comes directly from having high expectations is that teachers do not water down the mathematical content due to the preconceived notion that students are incapable of doing the mathematics. In other words, the key idea is that learning mathematics requires some level of challenge and effort that is only afforded if students work on hard tasks. If teachers always seek to make mathematics "easy", they actually rob their students of those opportunities to learn. Teachers should provide students the opportunity for to learn and demonstrate their ability to do the mathematics. By having high expectations, teachers are providing students with the opportunity to succeed at doing mathematics.

The next component of the Equity Principle is: "Equity requires accommodating differences to help everyone learn mathematics" (The Equity Principle). In an ever-changing society, mathematics teachers have to take in consideration the diversity seen in their classrooms. While maintaining high expectations, teachers need to make accommodations based upon the students' needs in their classroom. One means of accommodating to differences seen in the classroom is by teachers taking in consideration the prior knowledge students have upon entering their classroom. Students always learn mathematics based on their prior understandings rather than in a vacuum. Also, many students experience math in their everyday lives, but they often fail to make use of this in the classroom. Context and applications do not have to be a hindrance to learning, but they can be if the students are completely unfamiliar with the context being used. Not all students enter a mathematics classroom with the same mathematical ability or life experiences. Students obviously will not all be at the same mathematical ability, therefore, the teacher's responsibility is to provide any necessary assistance the student may need to gain understanding of mathematical concepts that the student was unable to understand previously. Assistance may come in the form of extra help before or after school and tutoring. For students that meet to the requirements for special education, mathematics teachers should work with the special education teachers in order to provide they needed support system for the students have the opportunity to succeed. Emphasis on serving students who are furthest behind in their mathematical understanding often hurts advanced students the most. In the case of these students, teachers should provide enrichment programs that allow for the students to further explore mathematical concepts at a more advanced level (The Equity Principle). Other than mathematical ability, students will have different life experiences that they bring to a mathematics class. Some of these differences in life experiences may include language and cultural differences. Teachers must be aware when students have a language barrier because this may mean that students are capable to perform at the desired mathematical ability, but may need assistance in making sense of language and conventions of mathematics. Cultural differences become an issue in mathematics when teachers refer to different contexts for word problems and other application problems. Teachers must be aware of students' backgrounds, so teachers know what references students are able to make when given some sort of applications problem. Teachers must be aware of these discrepancies in order to provide the best opportunity for students to learn and understand mathematics.

The final component of the Equity Principle is: Equity requires resources and support for all classrooms and all students" (The Equity Principle). The classroom teacher for all classrooms cannot easily ensure this last component of the equity principle, but a teacher can take steps in order to ensure that students are provided the necessary resources and support. A common resource being seen implemented in the classroom is technology. Technology, such as graphing calculators and computers, allows for teachers to demonstrate more examples that would otherwise take an extended period of time. With more examples, students are better able to see mathematical patterns, which in many cases will result in a better understanding of the concepts. Many schools do have access to technology, but some schools do not. For the schools that do not and for the schools that do have technology, teachers need to provide students with the necessary resources to be successful. Sometimes students need resources such as more mathematical background or a real life situation that helps students see the mathematical concept at work.

Students in essence need more tools so that they can recreate the mathematical concept without having to memorize formulas. The idea of more providing the students with more mathematical background or a real life situation will be further explored during the discussion of "meaningful understanding".

The goal of discussing equity was in order to fully understand the central question, which includes the word "access". Access in regards to the central question embodies what NCTM defines as equitable mathematics. As stated previously, equitable mathematics includes high expectations and worthwhile opportunities for all, accommodating differences to help everyone learn mathematics, and resources and support for all classrooms and all students. By meeting the NCTM's Equity Principle, students will be able to gain access to mathematical understanding For students to gain access, they must have a means of learning. Having the means to learn mathematics is vital in today's society. Mathematics acts as a 'gatekeeper' to economic success, active citizenship, and higher education in our society" (Wager, 83). This fact can be easily seen from assessment such as the SAT to gaining employment in STEM fields; mathematics is viewed as a basic measure of readiness. By not understanding mathematics, students are barred from a range of economic opportunities. As mentioned before, low expectations actually hinder students in their ability to learn. Not all students learn mathematics at the same pace or have the same background knowledge to reference back to, so teachers must be able to accommodate to the differences student bring to the classroom in order for students to have the opportunity to participate in the process of learning mathematics. Finally, students must have the resources and support needed, which may be the most important aspect of having a means of learning mathematics. As mentioned previously, teachers cannot guarantee that students have the technology or other materials in the classroom, but teachers can provide students with tools so that students can make connections to previous mathematical concepts and situations. When students have many tools to attack a particular problem, they will be able to recreate the situation in order to help them with the current problem at hand. Essentially, in order for students to gain access to mathematics, teachers must be teaching equitable mathematics as defined by NCTM. In other words teachers must be providing students with a means of learning mathematics.

Defining "Meaningful Understanding"

The next set of words that must be fully defined in order to understand the central questions is "meaningful understanding". When discussing "meaningful understanding", two primary viewpoints should be considered. The first of these viewpoints at its simplest form is commonly seen throughout mathematics, which begins with abstract mathematics and having students move into "applying" the mathematics to a situation. Many mathematics classes begin with learning mathematical concepts abstractedly, which means mathematics is being represented in terms of symbols and diagrams rather than words or activities. The student then sees how to apply this new learned abstract math to some situation in the form of a word problem. Part of the false lesson here is that the situation is not mathematics, and the abstract symbols are mathematics. This idea is not true, but it is the message we send through the language of "application." In many cases these word problems are contrived situations just made up for students to have a reason to apply the mathematics to some situation. I want to take this viewpoint and extend it a little bit so that the situations that students are applying the abstract mathematics are not contrived, but well-chosen situations. These situations should be situations that allow for students to see that the abstract mathematics is useful. The first viewpoint of "meaningful understanding" is that students begin in the abstraction and move to applicable mathematics, so the situation acts as a justification of the mathematics but is not really part of the mathematics itself.

The second viewpoint of "meaningful understanding" is where students begin with a situation in which some mathematical concept can be observed. This viewpoint then is moving from the situation to abstract mathematics. If the situation is experientially real to the students, then they can use their common sense to justify concepts pertinent to the mathematical concept. Not only do students have a way to justify the mathematics they are learning, but they will also have a point of reference. When students have this point of reference, they gain another tool that will assist them in solving later abstract problems because they can recreate the situation over again to help them understand the current problem. The second viewpoint of "meaningful understanding" is that students are able to justify their abstract mathematics by treating the mathematical concept as a model of the situation

One viewpoint is not more correct than the other. Both viewpoints can provide for students to have a meaningful understanding of mathematics. The first viewpoint allows for students to see that mathematics is useful, and the second viewpoint allows for student to have another resource to reference for future problems. If students are seeing mathematics as only abstract symbols, this is problematic in developing mathematical understanding. The ultimate goal is for students to be able to go back and forth between situational mathematics and abstract mathematics with ease because they understand that the same concepts are at play. Teachers will eventually want students to work mostly in the abstract mathematics realm due to efficiency it brings to doing mathematics, but situational mathematics provides students with a reason why they are learning the mathematics and another resource for sense-making.

To help illustrate when a student does not have a meaningful understanding of mathematics, I will use a famous mathematics education study "Benny's Conception of Rules and Answers in IPI Mathematics" (Erlwanger). The use of this study is to criticize the program being used but to demonstrate that Benny does not have a meaningful understanding of mathematics, and the program was based on completing mathematical objectives at the students own pace with limited to no teacher interaction. The student, Benny, was a student that had mastered the most objectives in the class. The interviewer doing this study began by asking Benny questions regarding fractions. The interviewer soon found out that some Benny's responses were correct but his understanding or justifications were not mathematically appropriate. In other instances Benny's responses were all together incorrect. Benny had developed a set of algorithms that assisted him in solving fraction problems. Examples of these algorithms for adding fractions are as follows:

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}, e.g., \frac{3}{10}+\frac{4}{10}=\frac{7}{10};$$

$$\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}, e.g., \frac{4}{3}+\frac{3}{4}=1;$$

$$\frac{a}{b}+\frac{c}{c}=1\frac{a}{b}, e.g., \frac{2}{3}+\frac{4}{4}=1\frac{2}{3};$$

$$\frac{a}{10}+\frac{b}{100}=\frac{a+b}{110}, e.g., \frac{6}{10}+\frac{20}{100}=\frac{26}{110}.$$

It can be easily seen that Benny's algorithms are not correct. Benny said at one point during the interview that he has many algorithms, one for each kind of problem. This demonstrates that Benny does not have a meaningful understanding of mathematics, or otherwise he might understand that changing the number in the fractions does not change the basic assumptions of adding fractions. Benny does not learn the rules when working with fractions or does not remember them. This relates to Benny learning without guidance and only working on practice problems until he could pass a skills check, which was the assumption of IPI instruction. Benny also does not have a situation for making sense of his solutions to test whether they are reasonable. Benny, therefore, does not have a meaningful understanding of mathematics, which is leading him to have a misconception about fractions.

Defining "Quantitative Reasoning"

To begin the process of understanding what quantitative reasoning is, we must start with defining what a quantity is. "A quantity is a conceived attribute of an object such that the individual envisions that the attribute admits a measurement process. A quantity is cognitively constructed—the quantities that people know are the quantities they have conceptualized" (Moore). In other words, a quantity is comprised of three aspects. The first is the quality, which is the type of measurement being used to measure an attribute of an object (Ellis). Qualities could be the distance of something or the time is takes for an event to occur. The second component of a quantity is the appropriate unit that must be assigned to the measurement being used (Ellis). If the quality were distance then an appropriate unit could be miles, feet, or inches depending on the situation. The final component of a quantity is the actual numerical value (Ellis). An example of this then would be having a circle as the object. The quality would be the distance around part of the circle or the arc length. An appropriate unit is radians, and the numerical value would be the number of radians for that specific arc length. Another aspect of quantities is to keep in mind the two different types of quantities. The first type of quantity is an extensive quantity, which can be directly measured such as distance and time. The other type of quantity is an intensive quantity such as speed. Speed consists of two separate extensive quantities formed by multiplication (Lobato). Now that the definition of quantity is better known, the definition of quantitative reasoning is much easier to understand. The definition of “quantitative reasoning is the process of analyzing a situation in terms of quantities and relationships among them” (Moore). In other words, quantitative reasoning is all about understanding the object at hand and the necessary attributes of that object that must be analyzed keeping in mind what defines a quantity. Quantitative reasoning therefore allows for students to gain a meaningful understanding because it involves students thinking of numbers as measures of some experientially real quantity rather than abstracted amounts.

Functions are central to modern mathematics curricula, especially at the secondary level. This contrast I have identified between abstracted numbers and quantities relating to situations provide different perspectives of what functions are and how they should be understood. Three such perspectives are quantitative reasoning, multi-representational, and phenomenological. The multi-representational perspective is that of one that “identifies conceptual understanding in the domain of mathematical functions as the set of connections that students make across representations, typically tables, equations, and graphs” (Lobato). The multi-representational perspective basically is that students are able to move between the different representations. The phenomenological perspective is one that is not as strictly defined, but is a perspective that takes in consideration the individual working on the mathematical task. In essence, the phenomenological perspective can be defined as a “reflective process in which meanings are developed within the social context of the participants and then re-interpreted as the participants reflect on their experiences and reorganize their conceptions” (Lobato). In other words, the phenomenological perspective is one that takes in consideration that every student will interpret a situation differently based on his or her experiences. These two perspectives are important to keep in mind because quantitative reasoning, multi-representational, and phenomenological perspectives are all related in some way. The best way to think about it is that the student works within his or her own experiences to develop a method to solving a problem, which may not always be the traditional way of thinking. The teacher then assists the student to examine the quantities and the relationships taking place between these quantities. Once the student understands the quantities, then he or she can represent the quantities in multiple ways. In other words, these three perspectives are at work during the whole process of learning.

Now that quantitative reasoning as been defined and other where it fits into the bigger picture, I will examine a few examples of quantitative reasoning will assist in seeing what constitutes quantitative reasoning in the mathematical curriculum. The first examples we will look at deal with linear functions and relationships. A study conducted by Dr. Amy Ellis exemplifies quantitative reasoning while learning the meanings behind linear functions and relationships. The study consisted of two different groups. One group was comprised of seven eighth grade algebra students, who primarily used contextless tables to develop generalizations about linear growth. The second group consisted of seven pre-algebra seventh graders, who encountered two real-world situations involving linearity. The two real-world situations were gear ratios and constant speed. The study found that the eighth-grade students focused more on numerical patterns, while the seventh-grade students focused on quantitative relationships.

Students who focused on patterns seemed to have more difficulty with dealing with situations that were not familiar to them. The study first begins to explain that the students would focus on patterns within a column of a table meaning all of the x-values of all of the y-values rather than look for relationships between corresponding (x, y) pairs. Due to students relying on this type of patterns, the study found that students would develop global rule about linearity that were dependent on uniform data tables. This means that students have to have a table that the x-value increases by a constant increment. The study found that because students had learned about linear functions in a classroom environment that emphasized one particular type of table, the uniform table, then student had much difficult justifying why a table that was not uniform would result in a linear relationship. This would be like saying the price of meat is only proportional to the weight being purchased if it was represented for each number of pounds (1, 2, 3, 4, etc.).

The study found that the seventh grade students, who focused on relationships were better able to explain what it meant for a set of data to have a linear relationship. The students had to begin with a real-world situation, and develop tables, which were not uniform. During the gear situation, students were given that the small gear had 8 teeth and the large gear had 12 teeth. The students then had to determine the relationship between rotations of each gear when they were connected to one another. The explanation by Dora exemplifies how she thinks about the situation quantitatively:

> Think of a gear...one gear has 8 teeth, the other has 12 teeth. When you spin them, teeth pass through each other. For every $\frac{2}{3}$ of the teeth passed on the large one, that's 8 teeth; the small one turns once. If the small one goes 3 turns, the large one will go 2. So if the small goes $7\frac{1}{2}$ times, the large gear will go 5. You can figure this out by setting it up in a fraction...5 over 7.5. Reduce it, and if it equals $\frac{2}{3}$, then it's right.

Dora's explanation includes the situation and involves the necessary quantities within the situation. Dora is able then to apply meaning to the number $\frac{2}{3}$, which is the ratio of the gears' rotations or the rate of change of the gear's rotations. The article notes that seventh-grade students were able to make generalizations about linearity by using the quantities in the situations. This suggests that working with quantitative relationships may help support student understanding of linearity as a mathematical relationship invariant across different situations or representations.

This study suggests that teachers should use quantitative reasoning in developing students' understanding. One way to do this is by developing situations that are experiential real such as the gear problem or the speed problem. The speed problem is very helpful because speed is something students experience and to which they can relate. Teachers can use a speed problem to help see changes in quantities and how the relationships between the quantities lead to a linear relationship. Besides developing experiential problems, the problems must also be meaningful. This means for example giving the first few terms in tabular form is not beneficial because in reality multiple functions can be derived from only having three pieces of data. The problems also must be realistic in order to be meaningful. An example in the study of a contrived linear relationship is the number of surfboards sold and the temperatures at the beach. This will not benefit students because a linear relationship between these two quantities would never exist. This situation is not realistic due to the fact the temperatures at the beach and the number of surfboards do not have a correlation to one another. This situation is not accessible to many students due to the fact many students do not surf. Teachers should avoid situations that are not realistic or accessible to students. Teachers should force students to speak quantitatively, instead of as x does this then y does this. Students should be expected to describe the relationship between two quantities using the actually quantitates in the situation. By implementing quantitative reasoning in the classroom, this study suggests that students will be able to understand relationship better and be able to make well-supported generalizations.

Throughout the semester I tried to find mathematical tasks that would lend students to have the ability to apply quantitative reasoning. One of these problems was called the "Barber Pole", and the problem is as follows: The revolving portion of a splendid barber pole is a cylinder 4 feet tall and 6 inches in radius. The red stripe is painted in a spiral around the cylinder and makes exactly 8 complete turns around it from bottom to top. How long is the stripe? (Ignore its width.) (Barber Pole).

In order to solve this problem, students must understand how the stripe behaves in relation to the cylinder. Students need to begin to think about the object in question and the important quantities. The object is a cylinder, and some important quantities are the height of the cylinder, the radius of the cylinder, and the number of rotations of the red stripe. To help students see how the red strips is behaving and how all three of these quantities relate, use a paper towel roll and first try to have students come to the conclusion that when a cylinder is "unwrapped" that the resulting shape is a rectangle. Now wrap a string from a point on the top base of the cylinder to a point on the bottom base so that the string completes one rotation. Have students predict what will result once the cylinder is opened up into a rectangle. The students will see that the string connects corners of the rectangle to form two right triangles. If students continue to have trouble, use the paper towel roll and draw a dark line on the preexisting lines that rotate around the paper towel roll. This will help students see that the number of right triangles is created depending on the number of complete rotations. Once students are able to see how the quantitates are relating to one another, the mathematical procedure is much easier. Students will only need to apply the Pythagorean theorem and account for all of the rotations.

What Can the Teacher Do?

Teachers can implement quantitative reasoning in the classroom because it helps to meet the Equity Principle. One of the components of the equity principle was to provide students with worthwhile opportunities to learn mathematics. Through the implementation of quantitative reasoning students are seeing that mathematics is part of the real-world and not a separate entity from everything else in life. Quantitative reasoning also gives students access to mathematical understanding by providing them with meaning for mathematical concepts. In many cases, students perceive high school mathematics as a science of manipulating symbols in very specific ways. These symbols may not actually be symbols to the student in the sense of symbolizing something else, but instead simply exist as markings on a page. Learning mathematics through situations grounds students' understanding in something other than symbols, which provides students another access point into mathematics. The logic of manipulating symbols can be derived from their sense-making in the situation that the symbols represent.

Quantitative reasoning also holds students to high standards due to the fact when using quantitative reasoning, students must be precise in the language they use. Students must define each quantity in the situation and then describe the situation using those quantities. Students are no longer able to give descriptions, such as x increases y increases. Quantitative reasoning pushes students to understand the situation and then begin to use their prior knowledge to understand a mathematical concept. Quantitative reasoning also provides accommodations for students because in order to implement quantitative reasoning, the situation must be realistic and accessible. In order for a situation to be realistic and accessible, teachers must understand the prior knowledge of the students. Another accommodation quantitative reasoning provides is that it does not solely rely on students being able to decipher the mathematical language, which often acts as an insurmountable barrier to student learning. By implementing quantitative reasoning in the classroom, teachers are helping to meet the Equity Principle on several fronts.

Other Equity Issues in Education and Barriers to Instructional Reform

There are other causes of inequity in mathematics education that will not be addressed through quantitative reasoning, and it is important to understand these other related factors. The first set of problematic issues concentrates on systematic problems that cannot really be controlled by the classroom teacher. A concept used in order to bring about more equitable mathematics was to end tracking students into different level of mathematics courses (Boaler, Silver). By ending tracking, students of all mathematical abilities will be placed into the same classroom and be taught the same material. There is evidence for the efficacy of such an instructional model, but it strongly depends upon the teachers' implementation. Many teachers are challenged to adapt lessons to the extent that top students can be sufficiently challenged while the least mathematically proficient students also have the means to learn and succeed. While tracking has been shown to have negative effects on student achievement by reifying students' lack of efficacy, the abolition of tracking poses other very real challenges, too. Another systematic problem is that of the administration. Some administrations and parents will view quantitative reasoning as not real mathematics, because it is not consistent with their image of mathematics instruction. Many interested parties have come to expect repetitive practice of abstract procedures. Teachers will need to keep this in mind and include quantitative reasoning practices when applicable so that other adults close to the teaching situation do not perceive that the teacher is abandoning important mathematics or lowering expectations for learning. Teachers should also explain to the administration that quantitative reasoning is a means to help students reach proficiency in procedural mathematics. Standardized testing may be another issue that teachers face in conjunction with teaching mathematics through quantitative reasoning. Teachers need remember as mentioned before that quantitative reasoning is a tool to help students understand mathematics. The ultimate goal it to have students use their understanding of quantitative relationships to operate with abstract symbols in a meaningful way, which is what is seen on standardized testing.

The other set of problematic issues are more directly linked to the teacher's actions. The first and foremost problem with implementing mathematics using quantitative reasoning is the time it takes to do so. The time that it takes to implement quantitative reasoning is primarily in regards to developing and picking situations that allow for students to implement quantitative reasoning and gain the desired outcome. Teachers can do this on their own or through collaboration, which leads to another possible problematic issue. Collaboration takes both time and willingness to work together. Teachers should be able to overcome differences and work together in order to help their students, but this does not always happen. In some cases, teachers just do not agree on the method that would best help students. The other aspect is time again. In the study, *A Multidimensional Mathematics Approach with Equitable Outcomes*, a school took many steps to help promote equity in mathematics. One of these methods was to develop a curriculum that provided the opportunity for students to work in groups to explore mathematical concepts. In order to develop these tasks and have the opportunity to reflect the tasks after implementation. Teachers met once a week throughout the school year and met for a week over several summers to develop problems (Boaler). This is a lot of time, which some teachers do not have or not willing to give unless being compensated. An issue then is whether the school deems the time and support for the development of these problems. The final possible problematic issue with implementing equitable mathematics is that discussion is very important in the process of learning (Boaler, Silver). Enticing students to take part in mathematical discussions requires creating a new classroom culture. The teacher must set the tone at the beginning of the year that the expectation of this class is that students will participate in discussion. The teacher must also provide the safe learning environment where student feel comfortable sharing their opinions. In reality, classroom culture and time constraints are the most prominent teacher specific problematic issues that must be addressed in order to implement quantitative reasoning.

Implementation

Common Core State Standards for Mathematics

To demonstrate the use of quantitative reasoning for developing a lesson that is experientially real for students, we are going to examine a representative concept in mathematics, which is rational expression and functions. We will discuss the steps in the process of implementing a lesson that includes the use of quantitative reasoning. The first aspect of implementing any new teaching strategy is to meet the Common Core State Standards for Mathematics (CCSSM). The CCSSM should be the backbone of every lesson plan because these are the standards that all teachers are supposes to achieve through instruction. For the concept of rational expressions and functions, we are going to focus on three CCSSM. The first of these is the following standard: CCSS.Math.Content.HSF-BF.A.1 "Write a function that describes a relationship between two quantities." The second standard is the following: CCSS.Math.Content.HSA-APR.D.6 "Rewrite simple rational expressions in different forms; write a(x)/b(x) in the form q(x) + r(x)/b(x), where a(x), b(x), q(x), and r(x) are polynomials with the degree of r(x) less than the degree of b(x), using inspection, long division, or, for the more complicated examples, a computer algebra system." The final standard is the following: CCSS.Math.Content.HSF-IF.C.7d (+) "Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior." As we discuss the lesson more in-depth, we will see where these standards are met.

Situation

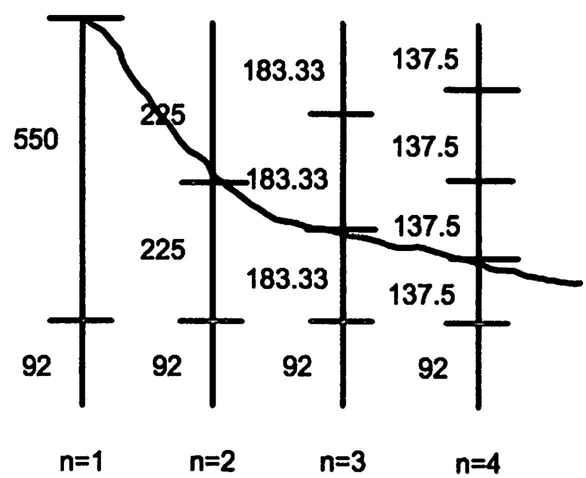
As previously discussed, the next step in implementing a lesson that uses quantitative reasoning is to have a well-chosen situation that allows for students to explore the desired mathematical concepts. The situation we will use is the initial cost of a refrigerator and the yearly electrical expenses resulting from that refrigerator (Compton). The important and relevant information for this situation is that the initial cost of the refrigerator is $500 and the yearly electrical expense is $92. The reason that this situation is a well-chosen situation is due to the fact it will allow for students to examine and interpret rational functions. This situation also provides the opportunity to discuss the meanings of the polynomial in the numerator and the polynomial in the denominator. In addition, this situation provides the ability to see that a ratio exists between this two polynomial, which is the rational function. This situation, in other words, allows for students to interpret each expression in terms of some meaningful aspect of the situation. This situation will also allow for students to rely on something they have some prior knowledge, which is money and time. This situation is simplified but not contrived, meaning that it is feasible and not created solely to fit the need for a mathematical example. Throughout the next sections we will see what key understandings can come from this one situation.

Key Understandings

The refrigerator situation is an example, which can be used in order to teach a few different concepts regarding rational expressions and functions. The first concept that would come from this situation is to write a function that gives the annual cost of the refrigerator as a function of the number of years you own the refrigerator. With the given information of the initial cost of the refrigerator being $500 and the yearly cost of the refrigerator's electricity being $92, the resulting function should be $C(n) = \frac{550+92n}{n}$. As part of quantitative reasoning we must relate each variable or expression to some quantity in the situation; C(n) is the average annual cost of the refrigerator and n is the number of years the refrigerator has been in service. Once again it is important to remember that this does help to meet the CCSSM of writing a function to describe the relation between two quantities, which in this example is the average annual cost of a refrigerator and the number of years the refrigerator has been in service.

Once the students have a function to examine and interpret, many other mathematical concepts may be discussed. The first of these is to understand the algebraic representation of each term of the function. The first term is 550, which is the initial cost of the refrigerator. The $92n$ is the cost of electricity dependent on the number of years the refrigerator has been owned. The term n in the denominator is the number of years the refrigerator has been owned. Therefore $550 + 92n$ results in the total cost of the refrigerator over n years. When dividing $550 + 92n$ by n the result is the average cost per year. For students to be able to explain what an algebraic function means in terms of the situation is important because the numbers and variables should each be interpreted in terms of the refrigerator situation. One of the CCSSM was to be able to rewrite rational functions in different forms. The function we have created can be rewritten by using algebraic rules. $C(n) = \frac{550+92n}{n} = \frac{550}{n} + \frac{92n}{n} = \frac{550}{n} + 92$. Student should know these algebraic rules before this point due to rational expressions usually being a later Algebra II concept. Once students have the function rewritten, they should be able to explain within the context of the situation why this new form of the function is still has meaning. If we recall, the function describes the average cost per year. The annual cost that remains constant from year to year is the cost of the electricity; therefore, the term that represents the annual cost of the electricity will remain constant. This explains why the 92 no longer has a variable attached to it because the 92 does not vary from year to year ("invariant"). The term $\frac{550}{n}$ does still contain a variable because the average annual cost of the refrigerator depends upon the number of years it is owned. We can easily see this by going through a couple of examples. When $n = 1$, then the average annual cost of the refrigerator is \$550. When $n = 2$, then the average annual cost of the refrigerator is \$225. When $n = 3$, then the average annual cost of the refrigerator is \$183.33. When $n = 4$, then the average annual cost of the refrigerator is \$137.5. When $n = 5$, then the average annual cost of the refrigerator is \$110. When $n = 20$, then the average annual cost of the refrigerator is \$27.50. Though we do not pay in this way, we are imagining paying equal amounts for the refrigerator every year it operates meaning we distribute the cost over *n* years. We see that the average annual cost of the refrigerator is decreasing each year. A decreasing average annual cost of the refrigerator should make sense because the individual paid for the refrigerator at a set price all at one time. As the years increase the average cost of the refrigerator will decrease due to imagining spreading the cost of the refrigerator out over a longer period of time. Student can actually use the concept of average cost to help them graph the rational function. This concept leads into the notion of "value", meaning that the refrigerator's value is relative to the number of years it runs. The importance of this concept is that it is applicable to everyday economics and is modeled in this mathematical context.

Before having the student graph the rational function through the use of a calculator or other more procedural means, the students should use the situation and the concept of averages to help them graph the function. Students know that the annual cost of electricity stays the same so that amount will remain constant from year to year. The average yearly cost of the refrigerator changes from year to year. By using the amount calculated previously, students can use those values to create a graph that models the relationship with also keeping the concept of averages in mind. The following diagram is an example of what student would create for the first four years of the relationship.



Once students have an understanding of the general behavior of the function from above, they can begin exploring an important attribute of the function, which are its asymptotes.

Students should explore the asymptotic behavior of this function within the context of the situation. The horizontal asymptote in this case is easier to see and explain within the situation. The function is as follows $C(n) = \frac{550}{n} + 92$. Student can begin by substituting different years into the function. Students will begin to see that the average cost per year begins to decrease. Student then should be asked what value is the function approaching as the number of years increase. Not all students will see this right away. By using the diagram they drew and by asking what are happening to the values of $\frac{550}{n}$ as n increases, students should see that the value of $\frac{550}{n}$ is becoming smaller and smaller as the number of years increase and as discussed before 92 remains constant. Students should come to the conclusion that the horizontal asymptote is $y = 92$ because no matter what value of n is substituted into the function, the average annual cost will never drop below \$92 due to the cost of the electricity. On the other hand, the vertical asymptote is not as easily seen from the function. The diagram can help, but values before year one have not really been discussed. To help students see what happens before the completion of one year, ask the students what the average annual cost would be if the refrigerator only worked for one day and then broke. The duration the refrigerator worked in terms of years would then be $n = \frac{1}{365}$. By substituting this into the function we find: $C(n) = \frac{550}{n} + 92, so\ C(\frac{1}{365}) = \frac{550}{\frac{1}{365}} + 92 = 365 * 550 + 92 = \$200{,}842$. At first glance, this number does not make any sense because the refrigerator only cost \$550 at the start. The teacher must help students see that the reason the average annual cost of the refrigerator is at such a large number is due to the fact we found the average rate after one day and then that rate would then be paid for the rest of the year, which increases the average annual cost of the refrigerator. In other words, if we had to pay \$550 each day of refrigerator service for a whole year, we would end up paying a lot of money. Another aspect of this large number demonstrates that as we become closer to the value $x = 0$, the function values get arbitrarily large (imagine if we paid \$550 every minute of refrigerator use). By demonstrating this, students will be able to see that that the vertical asymptote is $x = 0$. There are many aspects of rational functions that can be covered with this one situation. This situation allows for students to find meaning in algebraic expressions, graph rational functions using the idea of average rates, and see asymptotic behavior.

Assessment

Another key aspect of planning a lesson is to have a sound way of assessing students' understanding of the concepts covered during the lesson. The first topic covered was writing functions to describe the relationship between two quantities. To assess students understanding, the teacher can ask students to write a function that give the annual cost of a refrigerator a function of the number of years you own the refrigerator, but instead of the refrigerator initial cost being $550 it is now $1200. Students should then be asked what the steps they took to develop this function, what each part of the function means, and what the function as a whole means. This question then can lead into asking the students would this new refrigerator be worth the difference in cost if it were guaranteed to last at least 20 years. Students will have to demonstrate their ability to think about the average cost per year for each refrigerator. Students should be able to demonstrate that the $1200 refrigerator will have a higher average cost per year than the $550, but both will approach the same asymptote of $C = 92$.

Another set of problems should ask students to go back and forth between the situation and the function in regards of the asymptotic behavior. Let students consider the original problem where the initial cost of the refrigerator was $550 and the annual cost of electricity was $92. A question that should be asked of students is: What quantity would need to be changed in order to have a different horizontal asymptote? These question forces students to think about the quantities involved in the situation and how each affects the asymptotic behavior of the function. Student should also be asked to go in the other direction, which means if I changed the asymptote of the function, what new situation would be modeled? Students will have to work in the model of the situation and then move to the context. Asymptotes are a major concept when discussing rational functions, so by students being able to move easily between abstract and contextual situations asymptotes will help student gain a meaningful understanding of asymptotes.

The final set of questions that should be asked is in regards to the average cost per year of the refrigerator. A question that can be used is what would the average cost of the refrigerator be for $\frac{1}{3}$ of the year and explain what this value means within the context of the situation. Students can more easily comprehend the fact that has the years increase the average cost of the refrigerator decreases from year 1 to year infinity. Student however do have a difficult time conceptualizing the fact that if the refrigerator does not last a full year the average cost of the refrigerator will be greater than the initial cost of the refrigerator. This question forces students to first determine what the equivalent of $\frac{1}{3}$ of the year is, use the function to find the average cost of the refrigerator, and then explain what this value means. For this problem, $n = \frac{1}{3}$, so when substituted into the function we find: $C(n) = \frac{550}{n} + 92, so\ C\left(\frac{1}{3}\right) = \frac{550}{\frac{1}{3}} + 92 = \frac{3 * 550}{1} + 92 =$ \$1,742. We can think about this using ratio reasoning. If we paid \$550 for the initial cost of the refrigerator plus $\frac{\$92}{3}$ for electricity for $\frac{1}{3}$ of the year, then at that same rate we would have to spend $3 * \$550 + \$92$ for 1 year. We can see that this is true by the process of changing the duration for $\frac{1}{3}$ of the year to 1 year. In order to remain consistent, we also have to multiple each quantity of the cost by 3. Students will then have to explain that average annual cost of the refrigerator is above the initial cost due to the fact that at the moment the refrigerator broke the average rate after $\frac{1}{3}$ of the year was then paid for the remaining portion of the year, which makes the annual coast of the refrigerator more than the initial cost. In other words, the average annual cost is more than the initial cost because the cost per year rate was initially developed for a time interval less than one year. This means service at the same rate would actually increase for a full year. By having these questions that cover the main concepts taught during the lesson, the teacher will be able to see the level of understanding a student has of the concept. The goal of instruction is for students to gain a meaningful understanding of mathematics, so the questions should be aligned with the instructional objectives and meeting this goal.

Conclusion

The purpose of this project was to address the discrepancy between students of different socioeconomic status preparation in mathematics. The central question of the project was as follows: How can I as a teacher foster quantitative reasoning so that underrepresented students gain access to meaningful understanding of mathematics? In order to answer this question, key terms had to be defined first such as: equity and access, meaningful understanding, and quantitative reasoning. Once these terms were clearly defined, we saw that quantitative reasoning was a means to make mathematics education more equitable. Quantitative reasoning in essence provided students with another access point to mathematics. Once determining that quantitative reasoning was a feasible means to making mathematics more equitable for students, I took a mathematical concept, rational functions, and adapted a lesson that implemented quantitative reasoning. Through the implementation of quantitative reasoning, we saw that students had the opportunity to gain meaningful understanding of the mathematical concepts involved. Quantitative reasoning helps address some of the classroom-specific causes of inequity in mathematics education, thus providing a means for individual teachers to promote change through their practice. My future goals are to gain more experience in developing lessons implementing quantitative reasoning and implementing the lessons. We saw that there are some barriers to implementing quantitative reasoning, which were primarily time constraints and classroom culture, but these drawbacks were not insurmountable. I believe the benefits of implementing quantitative reasoning outweigh the drawback because students can be given access to mathematics, which in essence is access to future economic opportunities from which they would otherwise be barred.

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